The final exam will be on Laplace transforms and systems of differential equations.
The WeBWorK for Laplace transforms and systems should be part of your review.
The final exam will include 8 questions, one of which will be several True/False.
You will answer all questions (unlike Test 1, where you had choice).
You will have 120 minutes to complete the final exam.
You may use calculators, but you shouldn't need one to answer the questions.
You will be provided with at least the following information on the final exam:

Table of Elementary Laplace Transforms

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18a.	$f'(t)$	$sF(s) - f(0)$
18b.	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Problem 1. Answer as True or False:

$\mathcal{L}^{-1}[e^{-2s}\frac{1}{s-1}] = u_2(t)e^{t-2}$ .... T F

For $\delta(t)$ the the Dirac delta function, $\displaystyle\int_{-\infty}^{\infty} \delta(t-\pi/3)\sin t\, dt = \frac{\sqrt{3}}{2}$ .... T F

For $u_c(t)$ the Heaviside step function, $u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & -\infty < t < 5 \text{ and } t \ge 20 \end{cases}$ .... T F

An eigenvalue λ of matrix $\mathbf{A}$ with multiplicity k may have associated with it k linearly independent eigenvectors .. T F

If the point at $\mathbf{x} = \mathbf{0}$ is a stable center, the underlying physical system being modeled will have the property of conservation of energy .... T F

If the matrix $\mathbf{A}$ is complex-valued, its eigenvalues, if complex-valued, will still occur in complex-conjugate pairs ... T F

$W\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}\right] = -4e^{2t}$ .... T F

The vectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$, are linearly independent .... T F

Problem 2. Determine the Laplace transform of $f(t) = \sin at$ from the definition of the Laplace transform.

Problem 3. Using Laplace transforms, determine the solution to the initial value problem

$$y'' + 4y = \sin t + u_\pi(t)\sin(t-\pi), y(0) = 0, y'(0) = 0.$$

Hint: The partial fraction expansion you need will have the form $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4}$.

Problem 4.

(a) Calculate the Wronskian of the two vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$.

(b) Are the following vectors linearly independent? Show your calculations.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}.$$

Problem 5. Find the general solution of the following system of differential equations (you do not have to check linear independence):

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \mathbf{x}.$$

Problem 6. You are given the matrix $\mathbf{A}$ with eigenvalue λ of multiplicity 2, which has only one associated eigenvector

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \qquad \lambda = \lambda^{(1)} = \lambda^{(2)} = 2, \qquad \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A first solution of the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is therefore $\mathbf{x}^{(1)} = \xi e^{\lambda t}$. A second solution exists of the form $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} t + \eta e^{\lambda t}$. Determine this second solution by substituting into the system (show your work).

Problem 7. For the 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, discuss (include sketches) the classification and stability of the point $\mathbf{x} = \mathbf{0}$ based on the eigenvalues of the matrix $\mathbf{A}$. Which situation models a physical system which has the property of conservation of energy?

Problem 8. Use the method of undetermined coefficients to find a particular solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \qquad \mathbf{g}(t) = \begin{pmatrix} e^{3t} \\ e^{t} \end{pmatrix},$$

and the complementary solution is given by

$$\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Problem 9. Use the method of undetermined coefficients to find a particular solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \qquad \mathbf{g}(t) = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix},$$

and the complementary solution is given by

$$\mathbf{x}_c(t) = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Problem 10. Find the general solution of the following system of differential equations (you do not have to check linear independence):

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{x}.$$

Problem 11.

(a) Convert the following system of first order differential equations into a single second order differential equation.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(b) Are the following vectors linearly independent? Show your calculations.

$$\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 5 \\ -12 \end{pmatrix}.$$

Problem 12. Find the general solution of the following system of differential equations using eigenvalues and eigenvectors (you do not have to check linear independence):

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

Problem 13. You are given the matrix $\mathbf{A}$ with eigenvalue λ of multiplicity 2, which has only one associated eigenvector

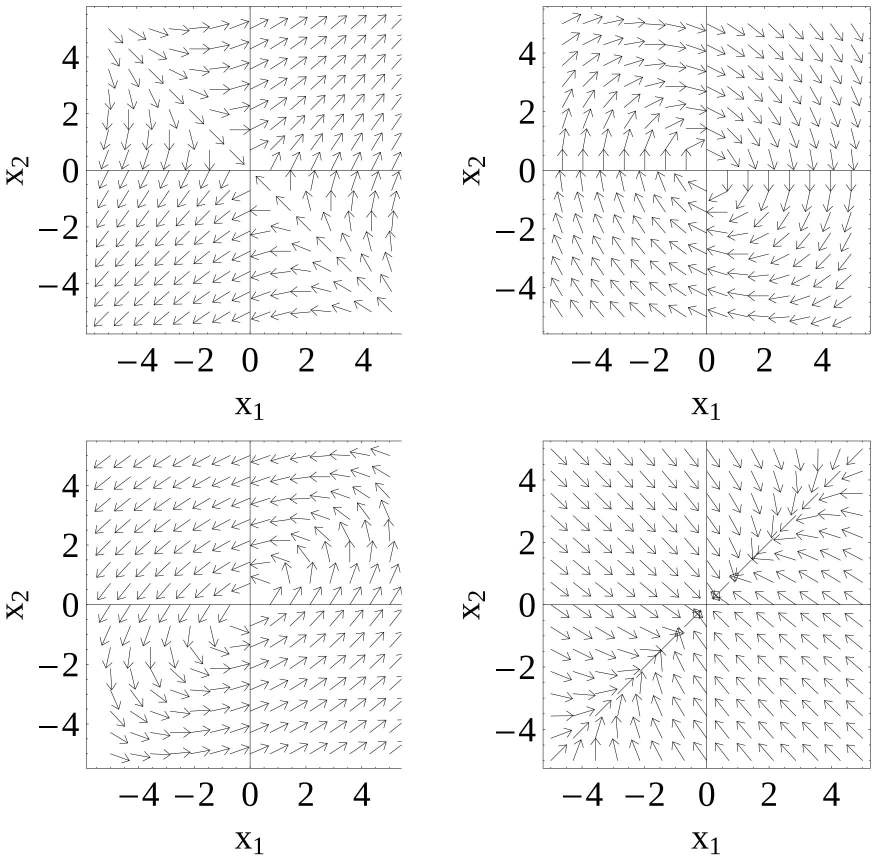
$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \qquad \lambda = \lambda^{(1)} = \lambda^{(2)} = 1, \qquad \xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A first solution of the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is therefore $\mathbf{x}^{(1)} = \xi e^{\lambda t}$. A second solution exists of the form $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} t + \eta e^{\lambda t}$. Determine this second solution by substituting into the system (show your work).

Problem 14. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{A}$ is a 2×2 matrix. Below are four direction fields associated with four different matrices $\mathbf{A}$.

(a) Sketch a solution curve in the x_1x_2-plane for each case. Classify the point $\mathbf{x} = \mathbf{0}$ as either stable or unstable.

(b) Explain what you know about the eigenvalues for the matrix $\mathbf{A}$ in each case.



15. Consider the nonlinear system:

$$x' = 2x + y^2$$
$$y' = 3y - x$$

Find the equilibrium point (x_0, y_0) that is not the origin. Classify the behaviour of the solution around the point. Sketch the behaviour of the solution to the nonlinear system around the equilibrium point. Make sure to include direction of motion in your sketch.