Second order linear equations with variable coefficients

General Form: $P(x)y'' + Q(x)y' + R(x)y = 0.$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \text{ for } |x - x_0| < \rho \neq 0, \qquad a_n = \frac{f^{(n)}(x_0)}{n!}$$

- how to calculate them for any function
- radius of convergence (ratio test, or distance to nearest complex pole)
- how to manipulate them (index of summation, power of x)

Example Determine the Taylor Series and radius of convergence of $f(x) = \dfrac{1}{(x-1)^2}$ about $x_0 = 0$.

$$\begin{aligned}
f(x) &= (x-1)^{-2} \\
f'(x) &= -1 \cdot 2(x-1)^{-3} \\
f''(x) &= 1 \cdot 2 \cdot 3(x-1)^{-4} \\
f'''(x) &= -1 \cdot 2 \cdot 3 \cdot 4(x-1)^{-5} \\
f^{(n)}(x) &= (-1)^n (n+1)!(x-1)^{-(n+2)} \\
f^{(n)}(x_0) = f^{(n)}(0) &= (-1)^n (n+1)!(-1)^{-(n+2)} = (n+1)! \\
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n
\end{aligned}$$

Radius of convergence: Use either complex poles or ratio test:

a) Complex Poles of $f(x)$:

$f(x)$ has a complex pole at $x = 1$. We are expanding about $x = 0$, so the distance to the nearest complex pole is 1. Therefore, $\rho = 1$, and we know our series will be absolutely convergent for $|x| < 1$.

b) Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)!\, x^{n+1}}{(n+1)!} \frac{n!}{(n+1)!\, x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+2)x}{(n+1)} \right| = |x| < 1 \text{ for absolute convergence}$$

The radius of convergence is $\rho = 1$.

How to classify points for a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$

- a function f is <u>analytic at x_0</u> if it has a Taylor series about $x = x_0$ which has a nonzero radius of convergence $\rho > 0$.
- a point x_0 is an ordinary point if
$$p(x) = \frac{Q(x)}{P(x)}, \qquad q(x) = \frac{R(x)}{P(x)}$$
are analytic at $x = x_0$. Otherwise x_0 is a singular point.
- a singular point x_0 is regular if
$$(x - x_0)p(x), \qquad (x - x_0)^2 q(x)$$
are analytic at $x = x_0$. Otherwise the point x_0 is an irregular singular point.

Example Find all the regular singular and irregular singular points of the differential equation

$$(x^2 - 4)^2 y'' + (x - 2)y' + y = 0.$$

First, identify $p(x)$ and $q(x)$:

$$p(x) = \frac{(x-2)}{(x^2-4)^2} = \frac{1}{(x-2)(x+2)^2}$$

$$q(x) = \frac{1}{(x^2-4)^2} = \frac{1}{(x-2)^2(x+2)^2}$$

The function $p(x)$ is not analytic at $x = +2$ and $x = -2$. Therefore, $x = \pm 2$ are singular points.
The function $q(x)$ is not analytic at $x = +2$ and $x = -2$. Therefore, $x = \pm 2$ are singular points.

Secondly, classify the singular points as regular or irregular:
<u>$x = +2$</u>:

$$(x-2)p(x) = \frac{1}{(x+2)^2}$$

$$(x-2)^2 q(x) = \frac{1}{(x+2)^2}$$

Both of these functions are analytic at $x = +2$. Therefore, $x = +2$ is a regular singular point.
<u>$x = -2$</u>:

$$(x+2)p(x) = \frac{1}{(x-2)(x+2)}$$

$$(x+2)^2 q(x) = \frac{1}{(x-2)^2}$$

The first function is not analytic at $x = -2$. Therefore, $x = -2$ is an irregular singular point.

Series solution about ordinary point x_0 for $P(x)y'' + Q(x)y' + R(x)y = 0$

Assume $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

- how to get the recurrence relation–substitute into the differential equation and manipulate (same power of x in each term, then each summation starting at same value of n) until you get

  $$\sum_{n=0}^{\infty} \textit{(recurrence relation)}(x - x_0)^n = 0$$

  which gives the *recurrence relation* $= 0$, $n = 0, 1, 2, 3, \ldots$.

  Note that sometimes the summation does not start at zero.

- remember to expand P, Q, R in power series about the same point!
- how to get a series solution from the recurrence relation
- estimate the minimum radius of convergence of the series solution (use Theorem 5.3.1, which says the series solutions will have radii of convergence at least as large as the radius of convergence for p, q)

Series solution about regular singular point

- Euler Equation

  $$x^2y'' + \alpha xy' + \beta y = 0$$

  - assume solution $y = x^r$, substitute into the differential equation to get the indicial equation:

    $$r(r - 1) + \alpha r + \beta = 0$$

  - solutions for distinct real roots

    $$y_1 = x^{r_1}, y_2 = x^{r_2}$$

  - repeated roots

    $$y_1 = x^{r_1}, y_2 = x^{r_1} \ln x$$

  - complex roots $r_{1,2} = \lambda \pm \mu i$

    $$y_1 = x^{\lambda} \cos(\mu \ln x), y_2 = x^{\lambda} \sin(\mu \ln x)$$

- General case

  $$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad \text{or} \qquad x^2y'' + x[xp(x)]y' + x^2q(x)y = 0$$

  about a regular singular point x=0.

  Assume a solution looks like: $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. You will be able to find at least one solution.

  - substitute into differential equation; leads to indicial equation (to determine r) and recurrence relation (to determine a_n).
  - roots of indicial equation are $r_1 \geq r_2$.

- how to get the series solution for r_1: substitute r_1 into the recurrence relation, then solve the recurrence relation for a_n.

$$y(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

- $r_1 \neq r_2$, and they do not differ by integer, you will be able to get two solutions.
- $r_1 = r_2$, use reduction of order to get second solution. NOT ON THIS TEST
- $r_1 - r_2 =$ integer, use reduction of order to get a second solution. NOT ON THIS TEST

Example Find the recurrence relations for the series solution about $x_0 = 1$ to $xy'' + y = 0$. Also, what will the minimum radius of convergence be for the series solutions that are ultimately obtained? You should not have to find the series solutions to answer the radius of convergence question.

Since $p(x) = 0$ and $q(x) = 1/x$ are both analytic at $x = 1$, the point $x = 1$ is an ordinary point. Therefore, assume

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n \\
y' &= \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}
\end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
xy'' + y &= 0 \\
(1 + (x-1))y'' + y &= 0 \\
(1 + (x-1)) \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x-1)^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=1}^{\infty} (n+1)(n)a_{n+1}(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n &= 0 \\
(2a_2 + a_0)(x-1)^0 + \sum_{n=1}^{\infty} \Big[(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + a_n\Big](x-1)^n &= 0
\end{aligned}$$

For this to be true for all values of x, we must have

$$\begin{aligned}
2a_2 + a_0 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + a_n &= 0, \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Since the first equation is the second equation with $n = 0$, we can write the recursion relations as

$$(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

Finally, since there is a complex pole of $q(x)$ at $x = 0$, and we are expanding about $x = 1$, the distance to the nearest complex pole is 1. Therefore, the radius of convergence of the Taylor series for $q(x)$ is $\rho = 1$. The minimum radius of convergence for the series solution will therefore be $\rho = 1$.

Example Find the recurrence relations and indicial equation for the series solution about $x_0 = 0$ to $xy'' + y = 0$. How many solutions do you expect to find by using the recursion relation method?

Since $q(x) = 1/x$ is not analytic at $x = 0$, the point $x = 0$ is a singular point.

Since $x^2 q(x) = x$ is analytic at $x = 0$, the point $x = 0$ is a regular singular point.

Since $p(x) = 0$ is analytic for all x, we don't need to worry about it when we classify the point.

Therefore, assume

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
xy'' + y &= 0 \\
x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=-1}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
r(r-1) a_0 x^r + \sum_{n=0}^{\infty} \Big[(n+r+1)(n+r) a_{n+1} + a_n \Big] x^{n+r} &= 0
\end{aligned}$$

For this to be true for all values of x, we must have

$$\begin{aligned}
r(r-1) a_0 &= 0 \\
(n+r+1)(n+r) a_{n+1} + a_n &= 0, \quad n = 0, 1, 2, 3, \ldots
\end{aligned}$$

Let's choose the first equation as the indicial equation, and we have $a_0 \neq 0$ and $r(r-1) = 0$ as the indicial equation. The recurrence relations are

$$(n+r+1)(n+r) a_{n+1} + a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

Since the roots of the indicial equation are $r_1 = 1$ and $r_2 = 0$, which differ by an integer, we only expect to be able to find a series solution using the recursion relations for the larger root, $r_1 = 1$. We could use reduction of order to get a second solution if necessary.

Example In solving a differential equation by a series method about $x = 0$, you arrive at the following indicial equation and recurrence relations:

$$r(r-1)a_0 = 0; \qquad (n+r+1)(n+r)a_{n+1} + a_n = 0, \;\; n = 0, 1, 2, 3, \ldots$$

Find one series solution these equations lead to.

We are only guaranteed to find a series solution if we use the larger of the two roots of the indicial equation. For this problem we have $r_1 = 1$ and $r_2 = 0$, so we choose $r_1 = 1$ to find the series solution.

The recurrence relations become

$$a_{n+1} = -\frac{a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3, \ldots$$

$$\begin{aligned} a_0 &= 1 \quad \text{arbitrary, not equal to zero} \\ a_1 &= -\frac{1}{(2)(1)} = -\frac{1}{1\cdot 2} \\ a_2 &= -\frac{a_1}{(3)(2)} = \frac{1}{1\cdot 2\cdot 2\cdot 3} \\ a_3 &= -\frac{a_2}{(4)(3)} = -\frac{1}{1\cdot 2\cdot 3\cdot 2\cdot 3\cdot 4} \\ a_4 &= -\frac{a_3}{(5)(4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 2\cdot 3\cdot 4\cdot 5} \end{aligned}$$

If we can't recognize a pattern, we write down the first few terms of the solution, which is

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = x^1 \left(-\frac{1}{1\cdot 2} + \frac{x}{1\cdot 2\cdot 2\cdot 3} - \frac{x^2}{1\cdot 2\cdot 3\cdot 2\cdot 3\cdot 4} + \frac{x^3}{1\cdot 2\cdot 3\cdot 4\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots \right)$$

Here we can recognize the pattern, so it is better to write the solution as

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n!(n+1)!}.$$

Laplace Transforms

- If f is a real-valued function defined on $[0, \infty)$, then the Laplace transform of f is defined as:

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

  - know how to work out the Laplace transform from the definition for common functions $f(t)$ that appear in Table 6.2.1.
  - Table 6.2.1 will be provided on the final exam.
  - Laplace transforms are useful when solving an initial value problem with discontinuous forcing function.

- The Heaviside step function is defined for $c \geq 0$ as

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

- The Dirac delta function $\delta(t)$ has the following properties:
  - $\delta(t - t_0) = 0$ for all $t \neq t_0$. It has an infinite spike at $t = t_0$.
  - $\displaystyle\int_a^b \delta(t - t_0) f(t)\, dt = \begin{cases} f(t_0) & \text{if } a \leq t_0 \leq b \\ 0 & \text{otherwise} \end{cases}$
  - $\mathcal{L}[\delta(t - t_0)] = e^{-st_0}$ for $t \geq 0$.

Example Solve this IVP using Laplace transforms: $y'' + 2y' + y = e^{-t} + 3\delta(t-1), \quad y(0) = 0, y'(0) = 0.$

Take Laplace transform:

$$\mathcal{L}\Big[y'' + 2y' + y = e^{-t} + 3\delta(t-1)\Big]$$

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[e^{-t}] + 3\mathcal{L}[\delta(t-1)]$$

$$\big(s^2 Y(s) - sy(0) - y'(0)\big) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s+1} + 3e^{-s}$$

$$s^2 Y(s) + 2sY(s) + Y(s) = \frac{1}{s+1} + 3e^{-s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{(s+1)^3} + 3e^{-s}\frac{1}{(s+1)^2}$$

Take Inverse Laplace transform:

$$\mathcal{L}^{-1}[Y(s)] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] + \mathcal{L}^{-1}\left[3e^{-s}\frac{1}{(s+1)^2}\right]$$

$$y(t) = \frac{1}{2}t^2 e^{-t} + 3u_1(t)\Big[(t-1)e^{-(t-1)}\Big]$$

Simplify:

$$y(t) = \begin{cases} \frac{1}{2}t^2 e^{-t} & \text{if } 0 \leq t < 1 \\ \frac{1}{2}t^2 e^{-t} + 3(t-1)e^{-(t-1)} & \text{if } t \geq 1 \end{cases}$$

Matrices and algebraic systems

- Matrices (multiplication, addition, etc)
- Algebraic Eigensystems (eigenvalues, eigenvectors, linear independence, Cramer's rule)

Systems of First Order Differential Equations

How these systems arise

- problems with several coupled dependent variables all of which are a function of a single independent variable
  - coupled spring-mass systems
  - predator-prey systems
  - Hamilton's equations for classical trajectory motion in a potential
- an nth order differential equation can be written as a system of linear differential equations

  $$mu'' + \gamma u' + ku = 0, \qquad u(0) = u_0, u'(0) = {u'}_0$$

  is equivalent to

  $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 \end{aligned}$$

  with initial conditions $x_1(0) = u_0, x_2(0) = {u'}_0$.

Basic Theory of Systems of First Order Linear Equations

- The general system of first order linear equations is of the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$
  - $\mathbf{x}$ is an n-vector
  - $\mathbf{P}(t)$ is an $n \times n$ matrix, in general time dependent
  - $\mathbf{g}(t)$ is an n-vector (the nonhomogeneous term)
- We choose this form since numerical methods, which are often used, assume this form.
- IVP will include n initial conditions $\mathbf{x}(0) = \mathbf{x}_0$.

Homogeneous Systems

- Equation is of the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$
- Solutions are written as

  $$\mathbf{x}^{(k)} = \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix}, \qquad 1 \leq k \leq n.$$

- Superposition principle
- Linear independence and the Wronskian
- Wronskian is the determinant of the fundamental matrix
- How this definition of the Wronskian relates to our previous definition
- fundamental set of solutions
- general solution

Nonhomogeneous Systems

- Equation is of the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$
- The general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$
- $\mathbf{x}_c$ is the complementary solution, found from solving the associated homogeneous problem.
- $\mathbf{x}_p$ is any particular solution of the nonhomogeneous equation. This can be found using one of three methods:
  1. Undetermined coefficients
  2. Variation of parameters (NOT ON TEST)
  3. Diagonalization (NOT ON TEST)

Homogeneous Systems with constant coefficients

- Equation is of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant $n \times n$ real valued matrix.
- Assume a solution of the form $\mathbf{x} = \xi e^{\lambda t}$, which upon substitution leads to solving the algebraic eigensystem $(\mathbf{A} - \lambda\mathbf{I})\xi = 0$.
- know how to get the general real-valued solution depending on if the eigenvalues are
  - Real and distinct
  - Real with multiplicity k, with $m < k$ associated linearly independent eigenvalues (for a 2×2 system, this would mean a real eigenvalue of multiplicity 2, with only one associated linearly independent eigenvalue)
  - complex
- Euler equation is of the form $t\mathbf{x}' = \mathbf{A}\mathbf{x}$, and we assume a solution of the form $\mathbf{x} = \xi t^{\lambda}$.

Stability

- know how to classify the point $\mathbf{x} = \mathbf{0}$ (and draw sketches to illustrate the behaviour of solutions) according to the eigenvalues of $\mathbf{A}$, and also discuss stability (asymptotically stable node, asymptotically unstable node, asymptotically unstable saddle point, etc (there were six cases we considered)).

Example Find the solution of the following initial value problem using eigenvalues and eigenvectors (you do not have to check linear independence):

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Also, find the explicit equation of the solution $x_2 = f(x_1)$ in the x_1x_2-plane.

Assume that $\mathbf{x} = \xi e^{\lambda t}$ is a solution, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\mathbf{x} = \xi e^{\lambda t}, \qquad \mathbf{x}' = \lambda \xi e^{\lambda t}$$

$$\text{Substitute:} \quad \lambda \xi e^{\lambda t} = \mathbf{A}\xi e^{\lambda t}$$

$e^{\lambda t} \neq 0$: $\quad (\mathbf{A} - \lambda I)\xi = \mathbf{0}$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ (1-\lambda)(1-\lambda) - 1 &= 0 \\ 1 - 2\lambda + \lambda^2 - 1 &= 0 \\ (\lambda - 2)\lambda &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = 0$ and $\lambda^{(2)} = 2$.

Eigenvectors:

$\lambda^{(1)} = 0$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\ (\mathbf{A} - 0I)\xi^{(1)} &= \mathbf{0} \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} \xi_1^{(1)} + \xi_2^{(1)} &= 0 \\ \xi_1^{(1)} + \xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(1)}$ to be arbitrary. Set $\xi_2^{(1)} = 1$.
Therefore, $\xi_1^{(1)} = -\xi_2^{(1)} = -1$.

$\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = 0$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{0t} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\lambda^{(2)} = 2$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(2)} I)\xi^{(2)} &= \mathbf{0} \\ (\mathbf{A} - 2I)\xi^{(2)} &= \mathbf{0} \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} -\xi_1^{(2)} + \xi_2^{(2)} &= 0 \\ \xi_1^{(2)} - \xi_2^{(2)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(2)}$ to be arbitrary. Set $\xi_2^{(2)} = 1$.
Therefore, $\xi_1^{(2)} = \xi_2^{(2)} = 1$.

$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = 2$.

A solution of the system of differential equations is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$.

<u>The General Solution</u>:

A general solution is therefore

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t)\mathbf{c}$$

The initial condition can be used to determine the constants:

$$\mathbf{x}(0) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which we can solve using Cramer's rule: $c_1 = \dfrac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \dfrac{-1}{-2} = \dfrac{1}{2}, \quad c_2 = \dfrac{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}}{-2} = \dfrac{-1}{-2} = \dfrac{1}{2}.$

The solution to the initial value problem is

$$\mathbf{x} = \begin{pmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 + e^{2t}/2 \\ 1/2 + e^{2t}/2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

To get the explicit function in the x_1x_2-plane, we need to eliminate the parameter t in the parametric representation $x_1 = -1/2 + e^{2t}/2$ and $x_2 = 1/2 + e^{2t}/2$. This can be done by solving for e^{2t}, which yields

$$2x_1 + 1 = e^{2t} = 2x_2 - 1 \longrightarrow x_2 = x_1 + 1.$$