Section 1.3

Determine the order of the given differential equation, also state whether the equation is linear or nonlinear.

Example (1.3.1) $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$

Second order; linear.

Example (1.3.2) $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

Second order; nonlinear. This is nonlinear due to the $y^2\dfrac{d^2y}{dt^2}$ term.

Example (1.3.3) $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

Fourth order; linear.

Example (1.3.4) $\dfrac{dy}{dt} + ty^2 = 0$

First order; nonlinear. This is nonlinear due to the y^2.

Example (1.3.5) $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin t$

Second order; nonlinear. This is nonlinear due to the $\sin(t+y)$, which in nonlinear in y.

Example (1.3.6) $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

Third order; linear.

Example (1.3.7) Verify that $y_1(t) = e^t$ and $y_2(t) = \cosh t$ are solutions of the differential equation $y'' - y = 0$.

To verify the solutions, we need to calculate derivatives and substitute into the differential equation.

$$\begin{aligned} y_1(t) &= e^t \\ y_1'(t) &= e^t \\ y_1''(t) &= e^t \end{aligned}$$

We have

$$y_1'' - y_1 = e^t - e^t = 0$$

Therefore, $y_1(t) = e^t$ satisfies the differential equation.

$$\begin{aligned} y_2(t) &= \cosh t \\ y_2'(t) &= \sinh t \\ y_2''(t) &= \cosh t \end{aligned}$$

We have

$$y_2'' - y_2 = \cosh t - \cosh t = 0$$

Therefore, $y_2(t) = \cosh t$ satisfies the differential equation.

Example (1.3.14) Verify that $y = e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}$ is a solution of the differential equation $y' - 2ty = 1$.

To verify the solution, we need to calculate derivatives and substitute into the differential equation.

$$
\begin{aligned}
y &= e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2} \\
y' &= \frac{d}{dt}\left[e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}\right] \\
&= e^{t^2} \frac{d}{dt}\left[\int_0^t e^{-s^2}\, ds\right] + \frac{d}{dt}\left[e^{t^2}\right] \int_0^t e^{-s^2}\, ds + \frac{d}{dt}\left[e^{t^2}\right] \\
&= e^{t^2}\left[e^{-t^2}\right] + \left[2te^{t^2}\right] \int_0^t e^{-s^2}\, ds + \left[2te^{t^2}\right] \\
&= 1 + 2te^{t^2} \int_0^t e^{-s^2}\, ds + 2te^{t^2}
\end{aligned}
$$

We have

$$
y'' - 2ty = 1 + 2te^{t^2} \int_0^t e^{-s^2}\, ds + 2te^{t^2} - 2t\left(e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}\right) = 1
$$

Therefore, $y = e^{t^2} \int_0^t e^{-s^2}\, ds + e^{t^2}$ satisfies the differential equation.