Section 2.1

Example (2.1.7) Draw a direction field for the differential equation:

$$\frac{dy}{dt} + 2ty = 2te^{-t^2}.$$

Based on the direction field, describe how the solutions behave for large values of t. Find the general solution of the given differential equation, and use it to determine how solutions behave as $\to \infty$.

The direction field analysis is contained in the *Mathematica* file.

To solve

$$\frac{dy}{dt} + 2ty = 2te^{-t^2}$$

we can use the integrating factor method. Multiply by a function $\mu = \mu(t)$:

$$\mu\frac{dy}{dt} + 2\mu ty = \mu 2te^{-t^2}.$$

Now, we want the following to be true:

$$\frac{d}{dt}[\mu y] = \mu y' + \mu' y \quad \text{(by the product rule)} \tag{1}$$
$$= \mu y' + 2\mu ty \quad \text{(the left hand side of our equation)} \tag{2}$$

Comparing Eqs. (1) and (2), we arrive at the differential equation that the integrating factor must solve:

$$2\mu t = \mu'.$$

This differential equation is separable, so the solution is

$$\begin{aligned}
2\mu t &= \frac{d\mu}{dt} \\
2t\,dt &= \frac{d\mu}{\mu} \\
\int 2t\,dt &= \int \frac{d\mu}{\mu} \\
t^2 &= \ln|\mu| + c_1 \\
e^{t^2}e^{-c_1} &= |\mu| \\
\mu &= c_2 e^{t^2}, \quad \text{where } c_2 = +e^{-c_1}
\end{aligned}$$

Therefore, the original differential equation becomes

$$\begin{aligned}
\mu\frac{dy}{dt} + 2\mu ty &= \mu 2te^{-t^2} \\
c_2e^{t^2}\frac{dy}{dt} + 2c_2e^{t^2}ty &= c_2e^{t^2}2te^{-t^2} \\
e^{t^2}\frac{dy}{dt} + 2e^{t^2}ty &= 2t \\
\frac{d}{dt}\left[e^{t^2}y\right] &= 2t
\end{aligned}$$

$$\begin{aligned} d\left[e^{t^2}y\right] &= 2t\,dt \\ \int d\left[e^{t^2}y\right] &= \int 2t\,dt \\ e^{t^2}y &= t^2 + c_3 \\ y(t) &= t^2 e^{-t^2} + c_3 e^{-t^2} \end{aligned}$$

Notice how important it is that we insert the constant of integration properly into our solution!

Now, for the large t limit, we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} t^2 e^{-t^2} + c_3 e^{-t^2} = 0$$

since the exponential decay term will dominate the t^2 part.

Example (2.1.18) Find the solution to the initial value problem $ty' + 2y = \sin t$, $y(\pi/2) = 1$.

To solve

$$ty' + 2y = \sin t$$

we can use the integrating factor method. We want the coefficient in front of the y' to be 1, so divide through by t before multiplying by a function $\mu = \mu(t)$:

$$\mu y' + \frac{2\mu}{t} y = \frac{\mu}{t} \sin t.$$

Now, we want the following to be true:

$$\begin{aligned} \frac{d}{dt}[\mu y] &= \mu y' + \mu' y \quad \text{(by the product rule)} \qquad (3) \\ &= \mu y' + \frac{2\mu}{t} y \quad \text{(the left hand side of our equation)} \qquad (4) \end{aligned}$$

Comparing Eqs. (3) and (4), we arrive at the differential equation that the integrating factor must solve:

$$\frac{2\mu}{t} = \mu'.$$

This differential equation is separable, so the solution is

$$\begin{aligned} \frac{2\mu}{t} &= \frac{d\mu}{dt} \\ \frac{2}{t}\,dt &= \frac{d\mu}{\mu} \\ \int \frac{2}{t}\,dt &= \int \frac{d\mu}{\mu} \\ 2\ln|t| &= \ln|\mu| + c_1 \\ \ln|t^2| &= \ln|\mu| + c_1 \\ e^{-c_1}|t^2| &= |\mu| \\ \mu &= c_2 t^2, \quad \text{where } c_2 = +e^{-c_1} \end{aligned}$$

Therefore, the original differential equation becomes

$$\begin{aligned}
\mu y' + \frac{2\mu}{t} y &= \frac{\mu}{t}\sin t \\
c_2 t^2 y' + \frac{2c_2 t^2}{t} y &= \frac{c_2 t^2}{t}\sin t \\
t^2 y' + 2ty &= t\sin t \\
\frac{d}{dt}\left[t^2 y\right] &= t\sin t \\
d\left[t^2 y\right] &= t\sin t\, dt \\
\int d\left[t^2 y\right] &= \int t\sin t\, dt \\
t^2 y &= \int t\sin t\, dt
\end{aligned}$$

This remaining integral can be done using parts:
Let $u = t, dv = \sin t\, dt$, so $du = dt, v = -\cos t$.

$$\begin{aligned}
\int t\sin t\, dt &= \int u\, dv \\
&= uv - \int v\, du \\
&= t(-\cos t) - \int(-\cos t)\, dt \\
&= -t\cos t + \sin t + c_3
\end{aligned}$$

Substituting back, we find

$$\begin{aligned}
t^2 y &= \int t\sin t\, dt \\
t^2 y &= -t\cos t + \sin t + c_3 \\
y &= -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{c_3}{t^2}
\end{aligned}$$

Now we can use the initial condition to determine the constant c_3:

$$\begin{aligned}
y(\pi/2) = 1 &= -\frac{\cos\pi/2}{(\pi/2)} + \frac{\sin\pi/2}{(\pi/2)^2} + \frac{c_3}{(\pi/2)^2} \\
1 &= 0 + \frac{1}{(\pi/2)^2} + \frac{c_3}{(\pi/2)^2} \\
(\pi/2)^2 &= 1 + c_3 \\
c_3 &= \frac{\pi^2}{4} - 1
\end{aligned}$$

The solution to the initial value problem is

$$y(t) = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{\pi^2}{4t^2} - \frac{1}{t^2}.$$

The solution is valid for $t > 0$.

Example (2.1.28) Consider the initial value problem $y' + 2y/3 = 1 - t/2$, $y(0) = y_0$. Find the value of y_0 for which the solution solution touches, but does not cross, the t-axis.

To solve

$$y' + \frac{2}{3}y = 1 - \frac{t}{2}$$

we can use the integrating factor method. Multiply by a function $\mu = \mu(t)$:

$$\mu y' + \frac{2\mu}{3}y = \mu - \frac{\mu t}{2}.$$

Now, we want the following to be true:

$$\begin{aligned} \frac{d}{dt}[\mu y] &= \mu y' + \mu' y \quad \text{(by the product rule)} && (5) \\ &= \mu y' + \frac{2\mu}{3}y \quad \text{(the left hand side of our equation)} && (6) \end{aligned}$$

Comparing Eqs. (5) and (6), we arrive at the differential equation that the integrating factor must solve:

$$\frac{2\mu}{3} = \mu'.$$

This differential equation is separable, so the solution is

$$\begin{aligned} \frac{2\mu}{3} &= \frac{d\mu}{dt} \\ \frac{2}{3}\,dt &= \frac{d\mu}{\mu} \\ \int \frac{2}{3}\,dt &= \int \frac{d\mu}{\mu} \\ \frac{2t}{3} &= \ln|\mu| + c_1 \\ e^{-c_1}e^{2t/3} &= |\mu| \\ \mu &= c_2 e^{2t/3}, \quad \text{where } c_2 = +e^{-c_1} \end{aligned}$$

Therefore, the original differential equation becomes

$$\begin{aligned} \mu y' + \frac{2\mu}{3}y &= \mu - \frac{\mu t}{2} \\ c_2 e^{2t/3} y' + \frac{2c_2 e^{2t/3}}{3}y &= c_2 e^{2t/3} - \frac{c_2 e^{2t/3} t}{2} \\ e^{2t/3} y' + \frac{2e^{2t/3}}{3}y &= e^{2t/3} - \frac{e^{2t/3} t}{2} \\ \frac{d}{dt}\left[e^{2t/3}y\right] &= e^{2t/3}\left(1 - \frac{t}{2}\right) \\ d\left[e^{2t/3}y\right] &= e^{2t/3}\left(1 - \frac{t}{2}\right)\,dt \\ \int d\left[e^{2t/3}y\right] &= \int e^{2t/3}\left(1 - \frac{t}{2}\right)\,dt \end{aligned}$$

$$\begin{aligned} e^{2t/3}y &= \int e^{2t/3}\left(1-\frac{t}{2}\right)\,dt \\ &= \frac{3}{2}e^{2t/3} - \frac{1}{2}\int e^{2t/3}t\,dt \end{aligned}$$

This remaining integral can be done using parts:
Let $u = t, dv = e^{2t/3}\,dt$, so $du = dt, v = \frac{3}{2}e^{2t/3}$.

$$\begin{aligned} \int e^{2t/3}t\,dt &= \int u\,dv \\ &= uv - \int v\,du \\ &= t(\frac{3}{2}e^{2t/3}) - \int (\frac{3}{2}e^{2t/3})\,dt \\ &= \frac{3t}{2}e^{2t/3} - \frac{9}{4}e^{2t/3} + c_3 \end{aligned}$$

Substituting back, we find

$$\begin{aligned} e^{2t/3}y &= \frac{3}{2}e^{2t/3} - \frac{1}{2}\int e^{2t/3}t\,dt \\ e^{2t/3}y &= \frac{3}{2}e^{2t/3} - \frac{1}{2}\left(\frac{3t}{2}e^{2t/3} - \frac{9}{4}e^{2t/3} + c_3\right) \\ y &= \frac{3}{2} - \frac{3t}{4} + \frac{9}{8} - \frac{c_3}{2}e^{-2t/3} \\ y &= \frac{21}{8} - \frac{3t}{4} - c_4 e^{-2t/3}, \quad \text{where } c_4 = c_3/2 \end{aligned}$$

Now we can use the initial condition to determine the constant c_3:

$$\begin{aligned} y(0) = y_0 &= \frac{21}{8} - c_4 \\ c_4 &= \frac{21}{8} - y_0 \end{aligned}$$

The solution to the initial value problem is

$$y(t) = \frac{21}{8} - \frac{3t}{4} - \left(\frac{21}{8} - y_0\right)e^{-2t/3}.$$

If this touches, but does not cross the t-axis, then the tangent line must be horizontal. Assume this happens at a point $\bar{t}$. Then we have:

$$y(\bar{t}) = 0 \text{ and } y'(\bar{t}) = 0.$$

We can solve these two equations for the two unknowns $\bar{t}$ and y_0.

$$y(\bar{t}) = 0 = \frac{21}{8} - \frac{3\bar{t}}{4} - \left(\frac{21}{8} - y_0\right)e^{-2\bar{t}/3}$$

$$y'(\bar{t}) = 0 = \frac{3}{4} + \frac{2}{3}\cdot\left(\frac{21}{8} - y_0\right)e^{-2\bar{t}/3}$$

These equations must be solved numerically. Using *Mathematica*, you can solve the second equation for $\bar{t}$ in terms of y_0, then substitute into the first equation and solve for y_0. You should find:

$$y_0 = -\frac{3}{8}\left(-7 + 3e^{4/3}\right) \sim -1.64288.$$

See the *Mathematica* file for the computational details.