Section 2.4

Example (2.4.15) Solve the initial value problem $y' + y^3 = 0$, $y(0) = y_0$ and determine how the interval in which the solution exists depends on the initial value y_0.

This is nonlinear since it has the term y^3. There may not be a general solution which contains all the solutions of the differential equation. We can still solve the differential equation, since it is separable.

$$\begin{aligned} \frac{dy}{dx} &= -y^3 \\ \frac{dy}{y^3} &= -dx \\ \int \frac{dy}{y^3} &= -\int dx \\ -\frac{y^{-2}}{2} &= -x + c \\ y^2 &= \frac{1}{2x - 2c} \end{aligned}$$

This is an implicit solution for y. The explicit solution is two functions:

$$y = +\frac{1}{\sqrt{2x - 2c}} \quad \text{and} \quad -\frac{1}{\sqrt{2x - 2c}}.$$

Apply the initial condition to determine the constant c. If $x = 0$, then $y = y_0$:

$$\begin{aligned} y_0^2 &= \frac{1}{0 - 2c} \\ c &= -\frac{1}{2y_0^2} \end{aligned}$$

The initial value problem has solution

$$y^2 = \frac{1}{2x + \frac{1}{y_0^2}} = \frac{y_0^2}{2xy_0^2 + 1}.$$

Remember the differential equation was nonlinear, so there may be other solutions not represented by this form. In this case, however, if $y_0 \in \mathbb{R}$ the function is defined, so we have all our solutions.

Frequently what is missing at this stage is some constant valued solutions, however, the only constant valued solution for this differential equation is $y(x) = 0$, which is contained in the above if $y_0 = 0$.

The interval of x for which this is valid requires the following to avoid taking the square root of a negative, and division by zero:

$$2xy_0^2 + 1 > 0 \longrightarrow x \in \left(-\frac{1}{2y_0^2}, \infty\right).$$

Notice that if $y_0 = 0$, the interval becomes $x \in (-\infty, \infty)$, which is what we would expect for a constant valued solution.

The *Mathematica* file contains some plots.

Example (2.4.21) Consider the initial value problem $y' = y^{1/3}, y(0) = 0$ where $t \geq 0$. Is there a solution that passes through $(1, 1)$? If so, find it. Is there a solution that passes though $(2, 1)$? If so, find it. Consider all possible solutions to the initial value problem. Determine the set of values that these solutions have at $t = 2$.

This problem is solved as Example 3 in the text. The solution is

$$y(t) = \begin{cases} 0 & 0 \leq t < t_0 \\ \pm[\frac{2}{3}(t - t_0)]^{3/2} & t \geq t_0 \end{cases}$$

For a solution to pass through $(1, 1)$, we require $y(1) = 1$ for some value of t_0.

$$y(1) = \pm[\frac{2}{3}(1 - t_0)]^{3/2} = 1 \text{ and } 1 \geq t_0$$

since if $1 < t_0$, then $y(1) = 0$. Solve the above for t_0.

$$\begin{aligned} \pm[\frac{2}{3}(1 - t_0)]^{3/2} &= 1 \\ \frac{2}{3}(1 - t_0) &= 1 \\ t_0 &= 1 - \frac{3}{2} = -\frac{1}{2} \end{aligned}$$

So $t_0 = -1/2 < 0$; but the problem states $t_0 \geq 0$. This contradiction tells us that no solution passes through $(1, 1)$.

For a solution to pass through $(2, 1)$, we require $y(2) = 1$ for some value of t_0.

$$y(2) = \pm[\frac{2}{3}(2 - t_0)]^{3/2} = 1 \text{ and } 2 \geq t_0$$

since if $2 < t_0$, then $y(2) = 0$. Solve the above for t_0.

$$\begin{aligned} \pm[\frac{2}{3}(2 - t_0)]^{3/2} &= 1 \\ \frac{2}{3}(2 - t_0) &= 1 \\ t_0 &= 2 - \frac{3}{2} = \frac{1}{2} \end{aligned}$$

So $t_0 = 1/2 > 0$, so there is a solution which passes through $(2, 1)$, and it is the one with $t_0 = 1/2$.

For all possible solutions at $t = 2$, we have:

$$y(2) = \begin{cases} 0 & 0 \leq 2 < t_0 \\ \pm[\frac{2}{3}(2 - t_0)]^{3/2} & 2 \geq t_0 \end{cases}$$

A clearer way to express this is as follows. If $t_0 > 2$, then $y(2) = 0$. If $t_0 \leq 2$, then $y(2) = \pm[\frac{2}{3}(2 - t_0)]^{3/2}$.

$$y(2) \in \left\{0, \pm[\frac{2}{3}(2 - t_0)]^{3/2}\right\} \text{ where } 0 \leq t_0 \leq 2.$$

Simplifying even further yields $|y| \leq (4/3)^{3/2}$.

More Detail about Example 2 in the Text: Example using Theorem 2.4.2 Solve the initial value problem:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \ y(0) = -1.$$

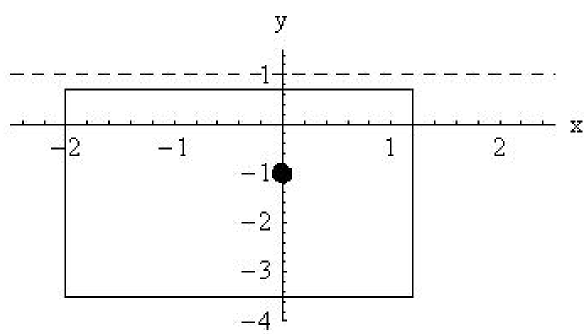
Note that this is nonlinear in both x and y. We must use Theorem 2.4.2.

Begin by identifying the quantities needed for Theorem 2.4.2:

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)}, \ \frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$$

Both these functions are continuous everywhere except when $y = 1$. Consequently, we **can** draw a rectangle about the initial point where both functions are continuous, and a unique solution will exist in this rectangle.

The size of this rectangle is not arbitrary, and we shall see that it there is a further constraint which is not yet apparent.



The important thing is we know a unique solution will exist in some rectangle about the initial point <u>before we solve the IVP</u>.

The DE is separable, so we shall proceed by writing it in the separable form:

$$2(y-1)dy = (3x^2+4x+2)dx, \tag{1}$$
$$\int 2(y-1)dy = \int (3x^2+4x+2)dx, \tag{2}$$
$$y^2 - 2y = x^3 + 2x^2 + 2x + C, \ \text{(implicit solution for } y(x)) \tag{3}$$
$$y = 1 \pm \sqrt{x^3+2x^2+2x+C+1}, \ \text{(explicit solution (using quadratic theorem))} \tag{4}$$

Here we do not stop at the implicit solution since we need to determine the rectangle for which the solution is valid. Apply the initial condition $y(0) = -1$:

$$y = 1 \pm \sqrt{x^3+2x^2+2x+C+1}, \tag{5}$$
$$-1 = 1 \pm \sqrt{0^3+2\cdot 0^2+2\cdot 0+C+1}, \tag{6}$$
$$-1 = 1 - \sqrt{C+1}, \ \text{(selects the minus sign)} \tag{7}$$
$$3 = C. \tag{8}$$
$$y = 1 - \sqrt{x^3+2x^2+2x+4}, \tag{9}$$

Now, we know the solution is bounded by the line $y = 1$. The square root also introduces a bound on x:

$$x^3 + 2x^2 + 2x + 4 \geq 0.$$

This cubic function has one real root, $x = -2$. For $x \geq -2$ the quantity under the square root will be positive, which is what we want. The final, unique solution is given by:

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}, \quad x \geq -2,\ y < 1.$$

So this solution will always be constrained to have $y < 1$ and $x \geq -2$, and it is unique (it is the only solution). The second solution to the differential equation does not exist when we apply the initial condition.

Notice also that the constraint $x \geq -2$ was not evident from the original differential equation. It was only discovered once we had solved the IVP.

A Different Initial Condition–loss of uniqueness of solution

If the initial condition had been $y(0) = 1$, we would not have been able to draw a rectangle around the point $(0, 1)$ for which f and $\partial f/\partial y$ would be continuous; we could not use Theorem 2.4.2 to say anything about the uniqueness or existence of the solution.

But this does **not** mean a solution does not exist! It just means we can't say anything about it to begin with. In fact, with the new initial condition, we have:

$$\begin{aligned}
y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x + C + 1}, && (10)\\
1 &= 1 \pm \sqrt{0^3 + 2 \cdot 0^2 + 2 \cdot 0 + C + 1}, && (11)\\
-1 &= C. && (12)\\
y &= 1 \pm \sqrt{x^3 + 2x^2 + 2x},\ x > 0 && (13)
\end{aligned}$$

and the solution exists, but is *not* unique. This is entirely due to the nonlinearity in the problem.