Section 2.8

Example (2.8.1) Transform the initial value problem $y' = t^2 + y^2$, $y(1) = 2$ into an equivalent problem with the initial point at the origin.

The transformation is driven by the initial condition. Define new coordinates:

$$\tilde{y}(\tilde{t}) = y(t) - 2, \tilde{t} = t - 1.$$

The initial conditions for the new coordinates become:

$$\begin{aligned} &\text{if } t = 1, \text{ then } \tilde{t} = 1 - 1 = 0. \\ &\tilde{y}(\tilde{t} = 0) = \tilde{y}(t = 1) = y(1) - 2 = 2 - 2 = 0, \end{aligned}$$

Start with the new equation:

$$\begin{aligned} \tilde{y}(\tilde{t}) &= y(t) - 2 \\ y(t) &= \tilde{y}(\tilde{t}) + 2 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}[y(t)] &= \frac{d}{dt}[\tilde{y}(\tilde{t}) + 2] \\ \frac{dy}{dt} &= \frac{d}{d\tilde{t}}\tilde{y}(\tilde{t}) \cdot \frac{d\tilde{t}}{dt} \\ &= \frac{d\tilde{y}}{d\tilde{t}} \cdot (1) \\ &= \frac{d\tilde{y}}{d\tilde{t}} \end{aligned}$$

The initial value problem therefore becomes:

$$\frac{d\tilde{y}}{d\tilde{t}} = (\tilde{t} + 1)^2 + (\tilde{y} + 2)^2, \tilde{y}(0) = 0.$$

Example (2.8.7) For the initial value problem $y' = ty + 1$, $y(0) = 0$, using Picard's method determine $\phi_n(t)$ and then plot $\phi_n(t)$ for $n = 1, 2, 3, 4$.

Picard's method provides approximate solutions to the initial value problem

$$y' = f(t, y), y(0) = 0,$$

by calculating the sequence of functions

$$\begin{aligned} \phi_0(t) &= 0 \\ \phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s))\, ds, n = 0, 1, 2, 3, 4, \ldots \end{aligned}$$

For this problem, $f(t, y) = ty + 1$ and we find:

$$\begin{aligned} \phi_0(t) &= 0 \\ \phi_1(t) &= \int_0^t f(s, \phi_0(s))\, ds = \int_0^t f(s, 0)\, ds = \int_0^t (s(0) + 1)\, ds = t \end{aligned}$$

$$
\begin{aligned}
\phi_2(t) &= \int_0^t f(s,\phi_1(s))\,ds = \int_0^t f(s,s)\,ds = \int_0^t (s(s)+1)\,ds = \frac{t^3}{3}+t \\
\phi_3(t) &= \int_0^t f(s,\phi_2(s))\,ds = \int_0^t f(s,\frac{s^3}{3}+s)\,ds = \int_0^t (s(\frac{s^3}{3}+s)+1)\,ds = \frac{t^5}{3\cdot 5}+\frac{t^3}{3}+t \\
\phi_4(t) &= \int_0^t f(s,\phi_3(s))\,ds = \int_0^t f(s,\frac{s^5}{3\cdot 5}+\frac{s^3}{3}+s)\,ds = \int_0^t (s(\frac{s^5}{3\cdot 5}+\frac{s^3}{3}+s)+1)\,ds = \frac{t^7}{3\cdot 5\cdot 7}+\frac{t^5}{3\cdot 5}+\frac{t^3}{3}+t
\end{aligned}
$$

From this, we can guess the pattern. It looks like we have

$$\phi_n(t) = \sum_{i=1}^{n} \frac{t^{2i-1}}{1\cdot 3\cdot 5\cdots(2i-1)}.$$

The plots are in the *Mathematica* file.

Example (2.8.14) Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}, 0 \le x \le 1$.

(a) Show that $\lim_{n\to\infty}\phi_n(x) = 0$ for $0 \le x \le 1$, and hence $\int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx = 0$.

(b) Show that $\int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$, and hence $\lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx = 1$.

In this example, $\lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx$ even though $\lim_{n\to\infty}\phi_n(x) = 0$ exists and is continuous.

(a)

$$
\begin{aligned}
\lim_{n\to\infty}\phi_n(x) &= \lim_{n\to\infty} 2nxe^{-nx^2} \\
&= 2x\lim_{n\to\infty}\frac{n}{e^{nx^2}} \longrightarrow \frac{\infty}{\infty} \text{ use l'Hospital's rule} \\
&= 2x\lim_{n\to\infty}\frac{\frac{d}{dn}[n]}{\frac{d}{dn}[e^{nx^2}]} \\
&= 2x\lim_{n\to\infty}\frac{1}{x^2e^{nx^2}} \\
&= \frac{2}{x}\lim_{n\to\infty}\frac{1}{e^{nx^2}} = 0 \text{ if } x \neq 0
\end{aligned}
$$

If $x = 0$, we have $\phi_n(0) = 2n(0)e^{-0} = 0$.

Therefore, $\lim_{n\to\infty}\phi_n(x) = 0$ for $0 \le x \le 1$.

(b)

$$
\begin{aligned}
\int_0^1 2nxe^{-nx^2}\,dx &= -\int_0^{-n} e^u\,du \text{ Substitution:} u = -nx^2; du = -2nx\,dx. \text{When } x = 1, u = -n; x = 0, u = 0. \\
&= e^u\big|_{-n}^{0} = 1 - e^{-n}
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\int_0^1 2nxe^{-nx^2}\,dx = \lim_{n\to\infty}(1 - e^{-n}) = 1$.