Section 2.9

Example (2.9.36) Find the general solution of the differential equation $t^2y'' + 2ty' - 1 = 0$.

Substitute $v = y'$; $v' = y''$. The differential equation becomes $t^2v' + 2tv - 1 = 0$, which is a first order differential equation. This method is called reduction of order (for obvious reasons) and we shall use it again in Section 3.5.

Solve this equation using the integrating factor technique:

Make sure the coefficient in front of the v' is 1 before you multiply by a function $\mu = \mu(t)$:

$$\mu v' + \frac{2\mu}{t}v = \frac{\mu}{t^2}.$$

Now, we want the following to be true:

$$\frac{d}{dt}[\mu v] = \mu v' + \mu' v \quad \text{(by the product rule)} \tag{1}$$

$$= \mu v' + \frac{2\mu}{t}v \quad \text{(the left hand side of our equation)} \tag{2}$$

Comparing Eqs. (1) and (2), we arrive at the differential equation that the integrating factor must solve:

$$\frac{2\mu}{t} = \mu'.$$

This differential equation is separable, so the solution is

$$\begin{aligned}
\frac{2\mu}{t} &= \frac{d\mu}{dt} \\
\frac{2dt}{t} &= \frac{d\mu}{\mu} \\
\int \frac{2dt}{t} &= \int \frac{d\mu}{\mu} \\
2\ln|t| &= \ln|\mu| + c_1 \\
\ln|t^2| &= \ln|\mu| + c_1 \\
|t^2| &= e^{c_1}|\mu| \\
\mu &= c_2t^2, \quad \text{where } c_2 = +e^{-c_1}
\end{aligned}$$

Therefore, the original differential equation becomes

$$\begin{aligned}
\mu v' + \frac{2\mu}{t}v &= \frac{\mu}{t^2} \\
c_2t^2v' + \frac{2c_2t^2}{t}v &= \frac{c_2t^2}{t^2} \\
t^2v' + 2tv &= 1 \\
\frac{d}{dt}[t^2v] &= 1 \\
d[t^2v] &= dt \\
\int d[t^2v] &= \int dt \\
t^2v &= t + c_3 \\
v &= \frac{1}{t} + \frac{c_3}{t^2}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{1}{t} + \frac{c_3}{t^2} \\
\int dy &= \int \left(\frac{1}{t} + \frac{c_3}{t^2} \right) dt \\
y &= \ln|t| - \frac{c_3}{t} + c_4
\end{aligned}$$

Example (2.9.42) Solve the differential equation $yy'' + (y')^2 = 0$.

This is nonlinear, so none of our techniques will work. We have to resort to the substitution discussed in the text.

Let $v = y' = \frac{dy}{dt}$. Then $y'' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} \cdot v$.

Substitute into the differential equation:

$$\begin{aligned}
yy'' + (y')^2 &= 0 \\
y\left(\frac{dv}{dy}v\right) + (v)^2 &= 0 \\
y\left(\frac{dv}{dy}\right) + v &= 0, \quad v \neq 0 \\
\int \frac{dv}{v} &= -\int \frac{dy}{y} \\
\ln|v| &= -\ln|y| + c_1 \\
\ln|v| &= \ln|1/y| + c_1 \\
v &= \frac{+e^{c_1}}{y} = \frac{c_2}{y}, \quad \text{where } c_2 = +e^{c_1}
\end{aligned}$$

Now, we have $v = y' = c_2/y$. Solve this differential equation:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{c_2}{y} \\
\int y\,dy &= \int c_2\,dt \\
\frac{y^2}{2} &= c_2 t + c_3
\end{aligned}$$

Therefore, $y^2 = k_1 t + k_2$ is an implicit solution to the differential equation.