Section 3.1

Example (3.1.1) Find the general solution of the differential equation $y'' + 2y' - 3y = 0$.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' + 2y' - 3y = 0 \longrightarrow (r^2 + 2r - 3)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 + 2r - 3 = (r+3)(r-1) = 0 \longrightarrow r_1 = -3, r_2 = +1$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i e^{r_i t} = c_1 e^{-3t} + c_2 e^t$$

Example (3.1.5) Find the general solution of the differential equation $y'' + 5y' = 0$.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' + 5y' = 0 \longrightarrow (r^2 + 5r)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 + 5r = r(r+5) = 0 \longrightarrow r_1 = 0, r_2 = -5$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i e^{r_i t} = c_1 + c_2 e^{-5t}$$

Example (3.1.7) Find the general solution of the differential equation $y'' - 9y' + 9y = 0$.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' - 9y' + 9y = 0 \longrightarrow (r^2 - 9r + 9)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 - 9r + r = 0$$

Use the quadratic formula to find the solution:

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{9 \pm \sqrt{81 - 36}}{2} \\ &= \frac{9 \pm 3\sqrt{5}}{2} \end{aligned}$$

Two solutions are

$$y_1(t) = \exp\left(\left[\frac{9 + 3\sqrt{5}}{2}\right] t\right), \quad y_2(t) = \exp\left(\left[\frac{9 - 3\sqrt{5}}{2}\right] t\right)$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 \exp\left(\left[\frac{9 + 3\sqrt{5}}{2}\right] t\right) + c_2 \exp\left(\left[\frac{9 - 3\sqrt{5}}{2}\right] t\right)$$

Example (3.1.9) Find the solution to the initial value problem $y'' + y' - 2y = 0, y(0) = 1, y'(0) = 1$. Sketch the solution and describe its behaviour as t increases.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' + y' - 2y = 0 \longrightarrow (r^2 + r - 2)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 + r - 2 = (r + 2)(r - 1) = 0 \longrightarrow r_1 = -2, r_2 = +1$$

Two solutions are

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{t}$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{-2t} + c_2 e^{t}$$

Use the initial conditions to determine the constants c_1 and c_2.

$$
\begin{aligned}
y(t) &= c_1 e^{-2t} + c_2 e^t \\
y'(t) &= -2c_1 e^{-2t} + c_2 e^t
\end{aligned}
$$

$$
\begin{aligned}
y(0) &= c_1 + c_2 = 1 \\
y'(0) &= -2c_1 + c_2 = 1
\end{aligned}
$$

We can use Cramer's rule to solve the two equations in two unknowns:

$$
c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{0}{3} = 0, \quad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}}{3} = \frac{3}{3} = 1.
$$

The initial value problem has solution $y(t) = e^t$. You can sketch this by hand. It is an exponential function, so it will increase exponentially as t increases.