Section 3.3

Example (3.3.1) Use Euler's formula to write e^{1+2i} in the form $a+ib$.

$$\begin{aligned} e^{1+2i} &= e^1 e^{2i} \\ &= e(\cos 2 + i\sin 2) \\ &= e\cos 2 + ie\sin 2) \end{aligned}$$

Example (3.3.3) Use Euler's formula to write $e^{i\pi}$ in the form $a+ib$.

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 \end{aligned}$$

Example (3.3.7) Find the general solution of the differential equation $y'' - 2y' + 2y = 0$.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' - 2y' + 2y = 0 \longrightarrow (r^2 - 2r + 2)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 - 2r + 2 = 0$$

Use the quadratic formula to find the solution:

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{4-8}}{2} \\ &= \frac{2 \pm 2\sqrt{-1}}{2} \\ &= 1 \pm i = \lambda \pm \mu i \end{aligned}$$

So $\lambda = 1$, $\mu = 1$. Two linearly independent solutions are

$$y_1(t) = e^{\lambda t}\cos\mu t = e^t \cos t, \quad y_2(t) = e^{\lambda t}\sin\mu t = e^t \sin t$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^t \cos t + c_2 e^t \sin t$$

Example (3.3.17) Find the solution of the initial value problem $y'' + 4y = 0, y(0) = 0, y'(0) = 1$.

This is a constant coefficient equation. Therefore, we assume that a solution of the form $y = e^{rt}$ exists.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$y'' + 4y = 0 \longrightarrow (r^2 + 4)e^{rt} = 0$$

Then r must be a root of the characteristic equation:

$$r^2 + 4 = 0 \longrightarrow r_{1,2} = \pm 2i = \lambda + \mu i.$$

So $\lambda = 0$, $\mu = 2$. Two linearly independent solutions are

$$y_1(t) = e^{\lambda t} \cos \mu t = \cos 2t, \quad y_2(t) = e^{\lambda t} \sin \mu t = \sin 2t$$

so the general solution is

$$y(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 \cos 2t + c_2 \sin 2t$$

Use the initial conditions to determine the constants c_1 and c_2.

$$\begin{aligned} y(t) &= c_1 \cos 2t + c_2 \sin 2t \\ y'(t) &= -2c_1 \sin 2t + 2c_2 \cos 2t \end{aligned}$$

$$\begin{aligned} y(0) &= c_1 = 0 \\ y'(0) &= +2c_2 = 1 \longrightarrow c_2 = 1/2 \end{aligned}$$

The initial value problem has solution $y(t) = \frac{1}{2} \sin 2t$. You can sketch this by hand. It is a sinusoid function, so it will increase oscillate with period π and amplitude $1/2$.