## Section 3.4

**Example (3.4.6)** Find a general solution of the differential equation $y'' - 6y' + 9y = 0$.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 6y' + 9y &= 0 \\ (r^2 - 6r + 9)e^{rt} &= 0 \\ r^2 - 6r + 9 &= 0 \quad \text{characteristic equation} \\ (r-3)^2 &= 0 \end{aligned}$$

The root of the characteristic equation is $r = 3$ of multiplicity 2.

A fundamental set of solutions is therefore

$$y_1 = e^{3t}, \quad y_2 = te^{3t}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{3t} + c_2 te^{3t}.$$

**Example (3.5.7)** Find a general solution of the differential equation $4y'' + 17y' + 4y = 0$.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 4y'' + 17y' + 4y &= 0 \\ (4r^2 + 17r + 4)e^{rt} &= 0 \\ 4r^2 + 17r + 4 &= 0 \quad \text{characteristic equation} \\ r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-17 \pm \sqrt{289 - 64}}{8} \\ &= -\frac{1}{4}, -4 \end{aligned}$$

The roots of the characteristic equation are $r_1 = -1/4$, and $r_2 = -4$.

A fundamental set of solutions is therefore

$$y_1 = e^{-t/4}, \quad y_2 = e^{-4t}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{-t/4} + c_2 e^{-4t}.$$

**Example (3.4.11)** Solve the initial value problem $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$. Sketch the graph of the solution and describe the behaviour of the solution as $t \to \infty$.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 9y'' - 12y' + 4y &= 0 \\ (9r^2 - 12r + 4)e^{rt} &= 0 \\ 9r^2 - 12r + 4 &= 0 \quad \text{characteristic equation} \\ r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{12 \pm \sqrt{144 - 144}}{18} \\ &= \frac{2}{3} \end{aligned}$$

The root of the characteristic equation is $r = 2/3$ of multiplicity 2.

A fundamental set of solutions is therefore

$$y_1 = e^{2t/3}, \quad y_2 = te^{2t/3}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{2t/3} + c_2 te^{2t/3}.$$

Use the initial conditions to determine the constants:

$$\begin{aligned} y(t) &= c_1 e^{2t/3} + c_2 te^{2t/3} \\ y'(t) &= c_1 \cdot \frac{2}{3} e^{2t/3} + c_2 e^{2t/3} + c_2 \cdot \frac{2}{3} te^{2t/3} \\ \\ y(0) &= 2 = c_1 \\ y'(0) &= -1 = c_1 \cdot \frac{2}{3} + c_2 \end{aligned}$$

The solution is $c_1 = 2$ and $c_2 = -7/3$.

The initial value problem has solution $y(t) = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$.

As $t \to \infty$ the solution decreases without bound, since $y'(t) = -\frac{1}{9}e^{2t/3}(9 + 14t) < 0$ for all values of $t > 0$ (remember, the first derivative tells you if a function is increasing or decreasing).

See the associated *Mathematica* file for a sketch.

**Example (3.4.16)** Solve the initial value problem $y'' - y' + y/4 = 0$, $y(0) = 2$, $y'(0) = b$.

Determine the critical value of $b$ that separates solutions that grow positively from those that eventually grow negatively.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - y' + y/4 &= 0 \\ (r^2 - r + 1/44)e^{rt} &= 0 \\ r^2 - r + 1/4 &= 0 \quad \text{characteristic equation} \\ r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1 \pm \sqrt{1-1}}{2} \\ &= \frac{1}{2} \end{aligned}$$

The root of the characteristic equation is $r = 1/2$ of multiplicity 2.

A fundamental set of solutions is therefore

$$y_1 = e^{t/2}, \quad y_2 = te^{t/2}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{t/2} + c_2 t e^{t/2}.$$

Use the initial conditions to determine the constants:

$$\begin{aligned} y(t) &= c_1 e^{t/2} + c_2 t e^{t/2} \\ y'(t) &= c_1 \cdot \frac{1}{2} e^{t/2} + c_2 e^{t/2} + c_2 \cdot \frac{1}{2} t e^{t/2} \\ \\ y(0) &= 2 = c_1 \\ y'(0) &= b = c_1 \cdot \frac{1}{2} + c_2 \end{aligned}$$

The solution is $c_1 = 2$ and $c_2 = b - 1$.

The initial value problem has solution $y(t) = 2e^{t/2} + (b-1)te^{t/2} = e^{t/2}[2 + (b-1)t]$.

If $b > 1$, the solution will grow positively as $t \to \infty$. If $b < 1$, the solution will grow negatively as $t \to \infty$. The value of $b$ which separates the two types of behaviour is $b = 1$.

See the *Mathematica* file for some sketches.

**Example (3.4.20)** Consider the differential equation $y'' + 2ay' + a^2 y = 0$. Show that one solution is $y_1(t) = e^{-at}$ by working through the characteristic equation solution. Then, use Abel's formula to show a second solution to the differential equation is $y_2(t) = te^{-at}$.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' + 2ay' + a^2 y &= 0 \\ (r^2 + 2ar + a^2)e^{rt} &= 0 \\ r^2 + 2ar + a^2 &= 0 \quad \text{characteristic equation} \\ (r+a)^2 &= 0 \end{aligned}$$

The root of the characteristic equation is $r = -a$ of multiplicity 2.

One solution of the differential equation is of the form $y_1(t) = e^{-at}$.

Abels' Theorem tells us $W(y_1, y_2)(t) = c_1 \exp\left(-\int p(t)\,dt\right)$. Therefore,

$$\begin{aligned} W(y_1, y_2)(t) &= c_1 \exp\left(-\int p(t)\,dt\right) \\ &= c_1 \exp\left(-\int 2a\,dt\right) \\ &= c_1 e^{-2at} \\ &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \quad \text{by definition} \end{aligned}$$

Take $y_1(t) = e^{-at}$, the solution we already know. Taking the derivative and substituting into the differential equation $c_1 e^{-2at} = vy_1(t)y_2'(t) - y_1'(t)y_2(t)$, we arrive at the following first order, linear differential equation in $y_2$:

$$y_2' + ay_2 = c_1 e^{-at}.$$

This can be solved using the integrating factor technique. Multiply by a function $\mu = \mu(t)$:

$$\mu y_2' + \mu a y_2 = \mu c_1 e^{-at}.$$

Now, we want the following to be true:

$$\begin{aligned} \frac{d}{dt}[\mu y_2] &= \mu y_2' + \mu' y_2 \quad \text{(by the product rule)} && (1) \\ &= \mu y_2' + \frac{2\mu}{3} y_2 \quad \text{(the left hand side of our equation)} && (2) \end{aligned}$$

Comparing Eqs. (1) and (2), we arrive at the differential equation that the integrating factor must solve:

$$\mu a = \mu'.$$

This differential equation is separable, so the solution is

$$\begin{aligned} \mu a &= \frac{d\mu}{dt} \\ \int a\,dt &= \int \frac{d\mu}{\mu} \\ \int a\,dt &= \int \frac{d\mu}{\mu} \\ at + c_2 &= \ln|\mu| \\ e^{at}e^{c_2} &= |\mu| \\ \mu &= c_3 e^{at} \quad \text{where } c_3 = +e^{c_2} \end{aligned}$$

Therefore, the original differential equation becomes

$$\begin{aligned} \mu y_2' + \mu a y_2 &= \mu c_1 e^{-at} \\ c_3 e^{at} y_2' + c_3 e^{at} a y_2 &= c_3 e^{at} c_1 e^{-at} \\ e^{at} y_2' + e^{at} a y_2 &= c_1 \\ \frac{d}{dt}\left[e^{at} y_2\right] &= c_1 \end{aligned}$$

$$\begin{aligned}
\int d\left[e^{at}y_2\right] &= \int c_1\, dt \\
e^{at}y_2 &= c_1 t + c_4 \\
y_2 &= c_1 t e^{-at} + c_4 e^{-at}
\end{aligned}$$

So a second solution is $y_2(t) = c_1 t e^{-at} + c_4 e^{-at}$. Since we are usually interested in a fundamental set of solutions, which will have no constants and no overlap between them, we can choose a fundamental set of solutions to be:

$$y_1(t) = e^{-at}, \text{ and } y_3(t) = te^{-at}.$$

This verifies the result we saw in class using a completely different method. We you can see things in two different ways that's a great thing! And on your assignment you will see a third way, which is very exciting indeed.

**Example (3.4.23)** Find a second solution of the differential equation $t^2y'' - 4ty' + 6y = 0$ given one solution is $y_1(t) = t^2$.

First, note that this is not a constant coefficient differential equation, so we cannot assume the solution looks like $y = e^{rt}$. In fact, this is an Euler equation, which we will study in Section 5.5. You might want to try to use a solution like $y = e^{rt}$ and see what goes wrong as you attempt to get the characteristic equation.

To use reduction of order, we assume a second soloution looks like $y = v(t)y_1(t) = vy_1 = vt^2$. When you know the second solution, it is a good idea to put it in right away.

$$\begin{aligned}
y(t) &= vt^2 \\
y'(t) &= v't^2 + 2vt \\
y''(t) &= v''t^2 + 4v't + 2v
\end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
t^2y'' - 4ty' + 6y &= 0 \\
t^2(v''t^2 + 4v't + 2v) - 4t(v't^2 + 2vt) + 6(vt^2) &= 0 \\
v''t^4 &= 0 \\
v'' &= 0 \quad \text{since } t > 0 \\
\frac{dv'}{dt} &= 0 \\
\int d[v'] &= 0 \\
v' + c_1 &= 0 \\
\frac{dv}{dt} &= -c_1 \\
\int dv &= -c_1 \int dt \\
v = c_2 - c_1 t
\end{aligned}$$

Sometimes you have to use the integrating factor technique to determine $v$.

A second solution to the differential equation is $y(t) = vt^2 = c_2t^2 - c_1t^3$.

From this we can identitify a fundamental set of solutions as $y_1(t) = t^2$ and $y_2(t) = t^3$.

A general solution is $y(t) = k_1t^2 + k_2t^3$.