Section 3.5

Example (3.5.3) Find a general solution of the differential equation $y'' - 2y' - 3y = -3te^{-t}$.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$y'' - 2y' - 3y = 0$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 2y' - 3y &= 0 \\ (r^2 - 2r - 3)e^{rt} &= 0 \\ r^2 - 2r - 3 &= 0 \quad \text{characteristic equation} \\ (r-3)(r+1) &= 0 \end{aligned}$$

The roots of the characteristic equation are $r_1 = 3$ and $r_2 = -1$. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{3t} + c_2 e^{-t}.$$

We can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous differential equation, since the form of $g(t) = -3te^{-t}$ is one of our special forms (polynomial and exponential).

Initially, assume a solution of the original differential equation is $Y(t) = (At + B)e^{-t}$, since $g(t)$ is the product of an exponential and a polynomial.

However, when we compare this $Y(t)$ with $y_c(t)$ we see that part of $Y(t)$ is contained in $y_c(t)$. This part will satisfy the homogeneous equation, and therefore not satisfy the nonhomogeneous equation! We know before we even start that this is not a solution of the nonhomogeneous differential equation! If we tried to substitute it in and determine the constants A and B, we would find that we are unable to determine A and B. You might want to try this to see what problems you run into.

Instead, consider $Y(t) = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}$. There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation. All we need to do is substitute it in and determine the value of the constants A and B.

$$\begin{aligned} Y(t) &= (At^2 + Bt)e^{-t} \\ Y'(t) &= (2At + B)e^{-t} - (At^2 + Bt)e^{-t} \\ Y''(t) &= (2A)e^{-t} - 2(2At + B)e^{-t} + (At^2 + Bt)e^{-t} \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} y'' - 2y' - 3y &= -3te^{-t} \\ (2A)e^{-t} - 2(2At + B)e^{-t} + (At^2 + Bt)e^{-t} - 2(2At + B)e^{-t} + 2(At^2 + Bt)e^{-t} - 3(At^2 + Bt)e^{-t} &= -3te^{-t} \\ (2A) - 2(2At + B) + (At^2 + Bt) - 2(2At + B) + 2(At^2 + Bt) - 3(At^2 + Bt) &= -3t \\ (2A - 2B - 2B) + (-4A + B - 4A + 2B - 3B)t + (A + 2A - 3A)t^2 &= -3t \\ (2A - 4B) + (-8A)t &= -3t \\ (2A - 4B) + (-8A + 3)t &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of the powers of t must be identically zero, which means $2A - 4B = 0$ and $-8A + 3 = 0$. Solving these two equations in the two unknowns A and B yields $A = 3/8$ and $B = 3/16$.

A particular solution is therefore $y_p(t) = Y(t) = (3t^2/8 + 3t/16)e^{-t}$.

A general solution to the nonhomogeneous differential equation is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right) e^{-t} \end{aligned}$$

Example (3.5.7) Find a general solution of the differential equation $2y'' + 3y' + y = t^2 + 3\sin t$.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$2y'' + 3y' + y = 0$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 2y'' + 3y' + y &= 0 \\ (2r^2 + 3r + 1)e^{rt} &= 0 \\ 2r^2 + 3r + 1 &= 0 \quad \text{characteristic equation} \\ (2r + 1)(r + 1) &= 0 \end{aligned}$$

The roots of the characteristic equation are $r_1 = -1/2$ and $r_2 = -1$. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{-t/2}$ and $y_2(t) = e^{-t}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{-t/2} + c_2 e^{-t}.$$

We can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous differential equation, since the form of $g(t) = t^2 + 3\sin t$ is one of our special forms (polynomial and sine).

We split the nonhomogeneous differential equation into two differential equations, and solve each in turn (since the differential equation is linear, we can add the two solutions we find to get the particular solution to the original nonhomogeneous differential equation).

$$2y'' + 3y' + y = t^2 \text{ and } 2y'' + 3y' + y = 3\sin t$$

Initially, assume a solution of the first differential equation is $Y_1(t) = At^2 + Bt + C$, since $g(t) = t^2$ is a polynomial.

There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation. All we need to do is substitute it in and determine the value of the constants A, B and C.

$$\begin{aligned} Y_1(t) &= At^2 + Bt + C \\ Y_1'(t) &= 2At + B \\ Y_1''(t) &= 2A \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} 2y'' + 3y' + y &= t^2 \\ 2(2A) + 3(2At + B) + (At^2 + Bt + C) &= t^2 \\ t^2(A-1) + t^1(6A + B) + t^0(4A + 3B + C) &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of the powers of t must be identically zero, which means $A - 1 = 0$, $= 0$ and $4A + 3B + C = 0$. Solving these three equations in the three unknowns A, B and C yields $A = 1$, $B = -6$ and $C = 14$.

A particular solution is therefore $Y_1(t) = t^2 - 6t + 14$.

Initially, assume a solution of the second differential equation is $Y_2(t) = A\sin t + B\cos t$, since $g(t) = 3\sin t$ is a sine.

There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation. All we need to do is substitute it in and determine the value of the constants A and B.

$$\begin{aligned} Y_2(t) &= A\sin t + B\cos t \\ Y_2'(t) &= A\cos t - B\sin t \\ Y_2''(t) &= -A\sin t - B\cos t \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} 2y'' + 3y' + y &= 3\sin t \\ 2(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + (A\sin t + B\cos t) &= 3\sin t \\ (-A - 3B - 3)\sin t + (3A - B)\cos t &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of the powers of t must be identically zero, which means $-A-3B-3 = 0$and $3A - B = 0$.

$$A = \frac{\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & -3 \\ 3 & -1 \end{vmatrix}} = -\frac{3}{10}, \qquad B = \frac{\begin{vmatrix} -1 & 3 \\ 3 & 0 \end{vmatrix}}{10} = -\frac{9}{10}.$$

A particular solution is therefore $Y_2(t) = -\frac{3}{10}\sin t - \frac{9}{10}\cos t$.

A general solution to the nonhomogeneous differential equation is

$$\begin{aligned} y(t) &= y_c(t) + Y_1(t) + Y_2(t), \\ &= c_1 e^{-t/2} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin t - \frac{9}{10}\cos t. \end{aligned}$$

Example (3.5.17) Find a general solution of the initial value problem $y'' + 4y = 3\sin 2t$, $y(0) = 2$, $y'(0) - 1$.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$y'' + 4y = 0$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' + 4y &= 0 \\ (r^2+4)e^{rt} &= 0 \\ r^2+4 &= 0 \quad \text{characteristic equation} \\ r &= \pm 2i \end{aligned}$$

The roots of the characteristic equation are $r_1 = +2i$ and $r_2 = -2i$, complex conjugates with $\lambda = 0$ and $\mu = 2$. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{\lambda t}\cos \mu t = \cos 2t$ and $y_2(t) = e^{\lambda t}\sin \mu t = \sin 2t$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 \cos 2t + c_2 \sin 2t.$$

We can use the method of undetermined coefficients to find a particular solution to the nonhomogeneous differential equation, since the form of $g(t) = 3\sin 2t$ is one of our special forms.

Initially, assume a solution of the second differential equation is $Y(t) = A\sin 2t + B\cos 2t$, since $g(t) = 3\sin 2t$ is a sine. This will not work since there is overlap with $y_c(t)$. Instead, choose $Y(t) = At\sin 2t + Bt\cos 2t$.

There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation. All we need to do is substitute it in and determine the value of the constants A and B.

$$\begin{aligned} Y(t) &= At\sin 2t + Bt\cos 2t \\ Y'(t) &= A\sin 2t + 2At\sin 2t + B\cos 2t - 2B\sin 2t \\ Y''(t) &= 4A\cos 2t - 4At\sin 2t - 4B\sin 2t - 4Bt\cos 2t \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} y'' + 4y &= 3\sin 2t \\ (4A\cos 2t - 4At\sin 2t - 4B\sin 2t - 4Bt\cos 2t) + 4(At\sin 2t + Bt\cos 2t) &= 3\sin 2t \\ (4A)\cos 2t + (-4B-3)\sin 2t &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of the sine and cosine must be identically zero, which means $4A = 0$and $-4B - 3 = 0$. Therefore, $A = 0, B = -3/4$.

A particular solution is therefore $Y(t) = -\frac{3}{4}t\cos 2t$.

A general solution to the nonhomogeneous differential equation is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t), \\ &= c_1\cos 2t + c_2\sin 2t - \frac{3}{4}t\cos 2t. \end{aligned}$$

Use the initial conditions to determine the constants c_1 and c_2:

$$\begin{aligned} y(t) &= c_1\cos 2t + c_2\sin 2t - \frac{3}{4}t\cos 2t \\ y'(t) &= -2c_1\sin 2t + 2c_2\cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t \end{aligned}$$

$$\begin{aligned} y(0) = 2 &= c_1 \\ y'(0) = -1 &= 2c_2 - \frac{3}{4} \end{aligned}$$

Therefore, $c_1 = 2$ and $c_2 = -1/8$. The solution to the initial value problem is

$$y(t) = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t.$$