Section 3.6

Example (3.6.1) Find a particular solution of the differential equation $y'' - 5y' + 6y = 2e^t$. Check your answer using undetermined coefficients.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$y'' - 5y' + 6y = 0$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 5y' + 6y &= 0 \\ (r^2 - 5r + 6)e^{rt} &= 0 \\ r^2 - 5r + 6 &= 0 \quad \text{characteristic equation} \\ (r-3)(r-2) &= 0 \end{aligned}$$

The roots of the characteristic equation are $r_1 = 3$ and $r_2 = 2$. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{3t} + c_2 e^{2t}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants (hence the name, variation of parameters):

$$Y(t) = \sum_{i=1}^{2} \mu_i(t) y_i(t) = \mu_1(t)e^{3t} + \mu_2(t)e^{2t}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)e^{3t} + 3\mu_1(t)e^{3t} + \mu_2'(t)e^{2t} + 2\mu_2(t)e^{2t} = 3\mu_1(t)e^{3t} + 2\mu_2(t)e^{2t}.$$

where we have introduced the condition:

$$\mu_1'(t)e^{3t} + \mu_2'(t)e^{2t} = 0. \tag{1}$$

Differentiate again:

$$Y''(t) = 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} + 9\mu_1(t)e^{3t} + 4\mu_2(t)e^{2t}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 5y' + 6y &= 2e^t \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} + 9\mu_1(t)e^{3t} + 4\mu_2(t)e^{2t} - 5(3\mu_1(t)e^{3t} + 2\mu_2(t)e^{2t}) + 6(\mu_1(t)e^{3t} + \mu_2(t)e^{2t}) &= 2e^t \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} &= 2e^t \qquad (2) \end{aligned}$$

All the terms without derivatives of the $\mu_i(t)$ should drop out at this stage; if they don't check your work! We have the two equations (Eq. (1) and (2)) in two unknowns:

$$\begin{aligned} \mu_1'(t)e^{3t} + \mu_2'(t)e^{2t} &= 0 \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} &= 2e^t \end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 0 & e^{2t} \\ 2e^t & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix}} = \frac{-2e^{3t}}{-e^{5t}} = +2e^{-2t},$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned} \mu_1'(t) &= 2e^{-2t} \\ \int d\mu_1(t) &= 2\int e^{-2t}\, dt \\ \mu_1(t) &= -e^{-2t} + k_1 = -e^{-2t} \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & 2e^t \end{vmatrix}}{\begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix}} = \frac{2e^{4t}}{-e^{5t}} = -2e^{-t},$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned} \mu_2'(t) &= -2e^{-t} \\ \int d\mu_2(t) &= -2\int e^{-t}\, dt \\ \mu_2(t) &= 2e^{-t} + k_2 = 2e^{-t} \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)e^{3t} + \mu_2(t)e^{2t} = -e^{-2t}e^{3t} + 2e^{-t}e^{2t} = e^t.$$

We can use the method of undetermined coefficients to check, since the form of $g(t) = 2e^t$ is one of our special forms.

Initially, assume a solution of the original differential equation is $Y(t) = Ae^t$, since $g(t)$ is an exponential. There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation.

All we need to do is substitute it in and determine the value of the constants A and B.

$$\begin{aligned} Y(t) &= Ae^t \\ Y'(t) &= Ae^t \\ Y''(t) &= Ae^t \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} y'' - 5y' + 6y &= 2e^t \\ Ae^t - 5Ae^t + 6Ae^t &= 2e^t \\ 2A &= 2 \\ A &= 1 \end{aligned}$$

A particular solution is therefore $y_p(t) = Y(t) = Ae^t = e^t$, which verifies what we found using variation of parameters.

Example (3.6.4) Find a particular solution of the differential equation $4y'' - 4y' + y = 16e^{t/2}$. Check your answer using undetermined coefficients.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$4y'' - 4y' + y = 0.$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 4y'' - 4y' + y &= 0 \\ (4r^2 - 4r + 1)e^{rt} &= 0 \\ 4r^2 - 4r + 1 &= 0 \quad \text{characteristic equation} \\ (2r-1)(2r-1) &= 0 \end{aligned}$$

The root of the characteristic equation is $r = 1/2$ of multiplicity two. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{t/2} + c_2 te^{t/2}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants:

$$Y(t) = \sum_{i=1}^{2} \mu_i(t) y_i(t) = \mu_1(t)e^{t/2} + \mu_2(t)te^{t/2}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)e^{t/2} + \mu_2'(t)te^{t/2} + \frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) = \frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right)$$

where we have introduced the condition:

$$\mu_1'(t)e^{t/2} + \mu_2'(t)te^{t/2} = 0. \tag{3}$$

Differentiate again:

$$Y''(t) = \frac{1}{2}\mu_1'(t)e^{t/2} + \mu_2'(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) + \frac{1}{4}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{4}te^{t/2} + e^{t/2}\right).$$

Substitute into the differential equation:

$$\begin{aligned} 4y'' - 4y' + y &= 16e^{t/2} \\ 4\left[\frac{1}{2}\mu_1'(t)e^{t/2} + \mu_2'(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) + \frac{1}{4}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{4}te^{t/2} + e^{t/2}\right)\right] & \\ -4\left[\frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right)\right] + \left[\mu_1(t)e^{t/2} + \mu_2(t)te^{t/2}\right] &= 16e^{t/2} \\ 2e^{t/2}\mu_1' + 4(e^{t/2} + \frac{t}{2}e^{t/2})\mu_2' &= 16e^{t/2} \qquad (4) \end{aligned}$$

All the terms without derivatives of the $\mu_i(t)$ should drop out at this stage; and they did! This is a good indication that our calculations are correct so far. We have the two equations (Eq. (3) and (4)) in two unknowns:

$$\begin{aligned} \mu_1'(t) + t\mu_2'(t) &= 0 \\ 2\mu_1'(t) + (4+2t)\mu_2'(t) &= 16 \end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 0 & t \\ 16 & 4+2t \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 4+2t \end{vmatrix}} = \frac{-16t}{4} = -4t,$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned} \mu_1'(t) &= -4t \\ \int d\mu_1(t) &= -4\int t\,dt \\ \mu_1(t) &= -2t^2 + k_1 = -2t^2 \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 4+2t \end{vmatrix}} = \frac{16}{4} = 4,$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned} \mu_2'(t) &= 4 \\ \int d\mu_2(t) &= 4\int dt \\ \mu_2(t) &= 4t + k_2 = 4t \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)e^{t/2} + \mu_2(t)te^{t/2} = -2t^2e^{t/2} + 4tte^{t/2} = 2t^2e^{t/2}.$$

We can use the method of undetermined coefficients to check, since the form of $g(t) = 16e^{t^2/2}$ is one of our special forms. I leave this to you.

Example (3.6.13) Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of the corresponding homogeneous differential equation for $t^2y'' - 2y = 3t^2 - 1$ for $t > 0$. Then find a particular solution of the nonhomogeneous differential equation.

This is a nonhomogeneous variable coefficient equation. We will learn how to solve variable coefficient differential equations using series solutions in Chapter 5. All we need to do is verify that the given functions are solutions.

Substitute y_1 into the differential equation:

$$\begin{aligned} y_1 &= t^2 \\ y_1' &= 2t \\ y_1'' &= 2 \\ t^2y'' - 2y &= t^2(2) - 2(t^2) \\ &= 0 \end{aligned} \tag{5}$$

Therefore, y_1 solves the associated homogeneous differential equation.

Substitute y_2 into the differential equation:

$$\begin{aligned} y_2 &= t^{-1} \\ y_2' &= -t^{-2} \\ y_2'' &= 2t^{-3} \\ t^2y'' - 2y &= t^2(2t^{-3}) - 2(t^{-1}) \\ &= 0 \end{aligned} \tag{6}$$

Therefore, y_2 solves the associated homogeneous differential equation.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1t^2 + c_2t^{-1}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants:

$$Y(t) = \sum_{i=1}^{2} \mu_i(t) y_i(t) = \mu_1(t)t^2 + \mu_2(t)t^{-1}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)t^2 + \mu_2'(t)t^{-1} + 2\mu_1(t)t - \mu_2(t)t^{-2} = 2\mu_1(t)t - \mu_2(t)t^{-2}$$

where we have introduced the condition:

$$\mu_1'(t)t^2 + \mu_2'(t)t^{-1} = 0. \tag{7}$$

Differentiate again:

$$Y''(t) = 2\mu_1'(t)t - \mu_2'(t)t^{-2} + 2\mu_1(t) + 2\mu_2(t)t^{-3}$$

Substitute into the differential equation:

$$\begin{aligned} t^2y'' - 2y &= 3t^2 - 1 \\ t^2\left[2\mu_1'(t)t - \mu_2'(t)t^{-2} + 2\mu_1(t) + 2\mu_2(t)t^{-3}\right] - 2\left[\mu_1(t)t^2 + \mu_2(t)t^{-1}\right] &= 3t^2 - 1 \\ 2t^3\mu_1' - \mu_2' &= 3t^2 - 1 \end{aligned} \tag{8}$$

We have the two equations (Eq. (7) and (8)) in two unknowns:

$$\begin{aligned} 2t^3\mu_1' - \mu_2' &= 3t^2 - 1 \\ \mu_1'(t)t^2 + \mu_2'(t)t^{-1} &= 0 \end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 3t^2-1 & -1 \\ 0 & t^{-1} \end{vmatrix}}{\begin{vmatrix} 2t^3 & -1 \\ t^2 & t^{-1} \end{vmatrix}} = \frac{3t - t^{-1}}{2t^2 + t^2} = t^{-1} - \frac{1}{3}t^{-3},$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned} \mu_1'(t) &= t^{-1} - \frac{1}{3}t^{-3} \\ \int d\mu_1(t) &= \int \left(t^{-1} - \frac{1}{3}t^{-3}\right) dt \\ \mu_1(t) &= \ln t + \frac{t^{-2}}{6} + k_1 = \ln t + \frac{t^{-2}}{6} \end{aligned}$$

Where we set the constant k_1 to zero since we are interested in any particular solution and there is no absolute value in logarithm since $x > 0$.

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} 2t^3 & 3t^2-1 \\ t^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2t^3 & -1 \\ t^2 & t^{-1} \end{vmatrix}} = \frac{-3t^4 + t^2}{2t^2 + t^2} = -t^2 + \frac{1}{3},$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned} \mu_2'(t) &= -t^2 + \frac{1}{3} \\ \int d\mu_2(t) &= \int \left(-t^2 + \frac{1}{3}\right) dt \\ \mu_2(t) &= -\frac{t^3}{3} + \frac{t}{3} + k_2 = -\frac{t^3}{3} + \frac{t}{3} \end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)t^2 + \mu_2(t)t^{-1} = \left(\ln t + \frac{t^{-2}}{6}\right)t^2 + \left(-\frac{t^3}{3} + \frac{t}{3}\right)t^{-1} = t^2\ln t + \frac{1}{2} - \frac{t^2}{3}.$$

Note that $-t^2/3 = cy_1(t)$ is actually a solution of the homogeneous equation. A simpler particular solution is therefore

$$y_p(t) = t^2 \ln t + \frac{1}{2}.$$

We could use the method of undetermined coefficients to check, since the form of $g(t) = 3t^2 - 1$ is one of our special forms.