Section 3.7

Example (3.7.1) Determine ω_0, R, and δ so $u = 3\cos 2t + 4\sin 2t = R\cos(\omega_0 t - \delta)$.

Let's work this through from first principles, rather than just using formulas.

$$\begin{aligned} u &= R\cos(\omega_0 t - \delta) \\ &= R\cos\omega_0 t\cos\delta + R\sin\omega_0 t\sin\delta \quad \text{(basic trig identity for cosine of a difference)} \\ &= 3\cos 2t + 4\sin 2t \end{aligned}$$

Comparing, we have

$$\begin{aligned} \omega_0 &= 2 \\ R\cos\delta &= 3 \\ R\sin\delta &= 4 \end{aligned}$$

A bit of algebra leads to

$$R^2\cos^2\delta + R^2\sin^2\delta = R^2 = 3^2 + 4^2 = 25 \longrightarrow R = 5,$$

$$\frac{R\sin\delta}{R\cos\delta} = \tan\delta = \frac{4}{3} \longrightarrow \delta = \arctan(4/3).$$

Therefore, $u = 3\cos 2t + 4\sin 2t = 5\cos(2t - \arctan(4/3))$.

Example (3.7.6) A mass of 100g stretches a spring 5cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping, determine the position of the mass at any time t. When does the mass first return to its equilibrium position?

We can use the equation of motion which was derived in class:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t).$$

where m is the mass, γ is the damping constant, k is the spring constant, $F(t)$ is the driving force, and $u(t)$ is the displacement.

No damping means $\gamma = 0$. No external force means $F(t) = 0$. The mass is $m = 1000\text{g}$. The spring constant is $k = mg/L = 1000\text{g} \times 980\text{cm/s}^2/5\text{cm} = 19600\text{g/s}^2$. Let's solve the differential equation, which is

$$mu''(t) + ku(t) = 0.$$

Since this is a constant coefficient differential equation, we can assume a solution looks like $u = e^{rt}$. Then

$$u = e^{rt}, \quad u' = re^{rt}, \quad u'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} mu''(t) + ku(t) &= 0 \\ (mr^2 + k)e^{rt} &= 0 \\ r^2 &= -\frac{k}{m} \end{aligned}$$

The mass and spring constant are both positive numbers, so r will be complex valued, $r = \pm\sqrt{k/m}\, i$. The roots of the characteristic equation are $r_1 = +\sqrt{k/m}\, i$ and $r_2 = -\sqrt{k/m}\, i$, complex conjugates with $\lambda = 0$ and $\mu = \sqrt{k/m}$.

A fundamental set of solutions to the associated homogeneous equation is $u_1(t) = e^{\lambda t} \cos \mu t = \cos \sqrt{k/m}t$ and $u_2(t) = e^{\lambda t} \sin \mu t = \sin \sqrt{k/m}t$. The solution to the differential equation, with t in seconds and u in cm, is

$$u(t) \sum_{i=1}^{2} c_i u_i(t) = c_1 \cos \sqrt{k/m}t + c_2 \sin \sqrt{k/m}t = c_1 \cos 14t + c_2 \sin 14t,$$

since $\sqrt{k/m} = 14\text{s}^{-1}$.

The initial conditions for this case are $u(0) = 0$ and $u'(0) = 10$ cm/s.

$$\begin{aligned} u(t) &= c_1 \cos 14t + c_2 \sin 14t \\ u'(t) &= -14c_1 \sin 14t + 14c_2 \cos 14t \end{aligned}$$

$$\begin{aligned} u(0) = 0 &= c_1 \\ u'(0) = 10 &= 14c_2 \end{aligned}$$

Therefore, $c_1 = 0$ and $c_2 = 5/7$.

The solution to the system is $u(t) = 5/7 \sin 14t$.

The first return to equilibrium is when $u(0) = 5/7 \sin 14t = 0$, or $\sin 14t = 0$. The mass is at equilibrium for $14t = \pi n$, $n = 0, 1, 2, 3, \ldots$.

$t = 0$: equilibrium position.

$t = \pi/14$: equilibrium position, velocity opposite sign of initial velocity.

$t = \pi/7$: equilibrium position, velocity same sign as initial velocity.

Example (3.7.11) A spring is stretched 10cm by a force of 3N. A mass of 2kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3N when the velocity of the mass is 5m/s. If the mass is pulled down 5cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position at any time t. Find the quasi frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

We can use the equation of motion which was derived in class:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t).$$

where m is the mass, γ is the damping constant, k is the spring constant, $F(t)$ is the driving force, and $u(t)$ is the displacement.

No external force means $F(t) = 0$. The mass is $m = 2$kg.

The spring constant is $k = mg/L = 3\text{N}/0.1\text{m} = 30\text{kg/s}^2$.

Viscous damping means $F_d = -\gamma u'(t)$, or $-3N = -\gamma 5\text{m/s}$, which yields $\gamma = 3/5$ kg/s.

The initial conditions are $u(0) = 0.05$m and $u'(0) = 0.1$m/s.

The initial value problem which models this situation is

$$2u''(t) + \frac{3}{5}u'(t) + 30u(t) = 0, u(0) = \frac{1}{20}, u'(0) = \frac{1}{10}.$$

Let's solve the differential equation, where t is in seconds and u in meters, which is

$$2u''(t) + \frac{3}{5}u'(t) + 30u(t) = 0.$$

Since this is a constant coefficient differential equation, we can assume a solution looks like $u = e^{rt}$. Then

$$u = e^{rt}, \quad u' = re^{rt}, \quad u'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 2u''(t) + \frac{3}{5}u'(t) + 30u(t) &= 0 \\ (2r^2 + 3r/5 + 30)e^{rt} &= 0 \\ 2r^2 + 3r/5 + 30 &= 0 \end{aligned}$$

The roots are complex, $r = -3/20 \pm \sqrt{5991}/20\, i$.

The solution to the differential equation, with t in seconds and u in cm, is

$$u(t) \sum_{i=1}^{2} c_i u_i(t) = c_1 e^{-3t/20} \cos \sqrt{5991}t/20 + c_2 e^{-3t/20} \sin \sqrt{5991}t/20.$$

Let's, for a change, use *Mathematica* to solve for the constants using the initial conditions (it would be tedious to write out by hand). The *Mathematica* file contains the details. We find $c_1 = 1/20$ and $c_2 = 43/(20\sqrt{5991})$.

The solution to the initial value problem is

$$u(t) = \frac{1}{20} e^{-3t/20} \cos\left(\frac{\sqrt{5991}t}{20}\right) + \frac{43}{20\sqrt{5991}} e^{-3t/20} \sin\left(\frac{\sqrt{5991}t}{20}\right).$$

To get the quasi-frequency, we need to identify the ω_0. Referring to the results from Problem 3.8.1, we can easily identify $\omega_0 = \frac{\sqrt{5991}}{20}$. The quasi-frequency is therefore

$$\mu = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \omega_0 = \left(1 - \frac{(3/5)^2}{4(30)(2)}\right)^{1/2} \frac{\sqrt{5991}}{20} = \frac{1997\sqrt{3/5}}{400} \sim 3.86717\text{rad/s}.$$

The ratio of the quasi-frequency to the natural frequency is

$$\frac{\mu}{\omega_0} = \frac{3997}{4000} \sim 0.99925.$$