Section 4.1: General Theory of nth Order Linear Equations

Example (4.1.4) Determine the intervals in which solutions are sure to exist for the differential equation $y''' + ty'' + t^2y' + t^3y = \ln t$.

The functions $P_1(t) = t$, $P_2(t) = t^2$ and $P_3(t) = t^3$ are continuous on $t \in (-\infty, \infty)$.

$g(t) = \ln t$ is continuous on $t \in (0, \infty)$.

Therefore, a solution exists on the interval $I = (0, \infty)$.

Example (4.1.6) Determine the intervals in which solutions are sure to exist for the differential equation $(x^2 - 4)y^{(iv)} + x^2y''' + 9y = 0$.

Rewrite in standard form: $y^{(iv)} + \dfrac{x^2}{x^2-4}y''' + \dfrac{9}{x^2-4}y = 0$.

The functions $P_3(t) = \dfrac{x^2}{x^2-4}$, $P_6(t) = \dfrac{9}{x^2-4}$ are continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

Therefore, solutions are sure to exist on the interval $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

Example (4.1.7) Determine whether the functions $f_1(t) = 2t - 3$, $f_2(t) = t^2 + 1$, and $f_3(t) = 2t^2 - t$ are linearly independent or linearly dependent. If they are linearly dependent, find a relation between them.

$$
\begin{aligned}
W(f_1, f_2, f_3)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} \\
&= \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix} \\
&= (2t-3)\begin{vmatrix} 2t & 4t-1 \\ 2 & 4 \end{vmatrix} - (2)\begin{vmatrix} t^2+1 & 2t^2-t \\ 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} t^2+1 & 2t^2-t \\ 2t & 4t-1 \end{vmatrix} \\
&= (2t-3)(\cancel{8t} - \cancel{8t} + 2) - (2)(\cancel{4t^2} + 4 - \cancel{4t^2} + 4t) \\
&= 4t - 6 - 8 - 8t \\
&= -4t - 14 \neq 0 \text{ for } t = 0
\end{aligned}
$$

Therefore, $2t - 3$, $t^2 + 1$, and $2t^2 - t$ are linearly independent.

Example (4.1.11) Verify the functions $f_1(t) = 1$, $f_2(t) = \cos t$, and $f_3(t) = \sin t$ are solutions of the differential equation $y''' + y' = 0$. Determine their Wronskian.

Verify they are solutions via direct substitution:

$$y_1(t) = 1, \quad y_1'(t) = 0, \quad y_1''(t) = 0, \quad y_1'''(t) = 0.$$

$$y''' + y' = 0 + 0 = 0.$$

$$y_2(t) = \cos t, \quad y_2'(t) = -\sin t, \quad y_2''(t) = -\cos t, \quad y_2'''(t) = \sin t.$$

$$y''' + y' = \sin t + (-\sin t) = 0.$$

$$y_3(t) = \sin t, \quad y_3'(t) = \cos t, \quad y_3''(t) = -\sin t, \quad y_3'''(t) = -\cos t.$$

$$y''' + y' = -\cos t + (\cos t) = 0.$$

$$\begin{aligned} W(1, \cos t, \sin t)(t) &= \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} \\ &= (1) \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} + 0 \\ &= \sin^2 t + \cos^2 t = 1 \end{aligned}$$

Therefore, 1, $\cos t$, and $\sin t$ are linearly independent.