Section 4.2: Homogeneous Equations with Constant Coefficients

Examples (4.2.1) (4.2.3) and (4.2.4) See *Mathematica* file.

Example (4.2.8) Find the square root of the complex number $1-i$.

From the *Mathematica* file we see that we have $1-i=\sqrt{2}\exp(i(7\pi/4+2\pi m))$, $m=\ldots,-2,-1,0,1,2,\ldots$.

$$\begin{aligned}
(1-i)^{1/2} &= \left(\sqrt{2}\exp(i(7\pi/4+2\pi m))\right)^{1/2}\\
&= 2^{1/4}\exp(i(7\pi/8+\pi m))\\
&= 2^{1/4}\exp(i(7\pi/8+\pi(0))) \text{ or } 2^{1/4}\exp(i(7\pi/8+\pi(1))) \qquad \text{other } m \text{ produce these same two values}\\
&= 2^{1/4}\exp(7\pi i/8) \text{ or } 2^{1/4}\exp(15\pi i/8)\\
&= 2^{1/4}\left(\cos(7\pi/8)+i\sin(7\pi/8)\right) \text{ or } 2^{1/4}\left(\cos(15\pi/8)+i\sin(15\pi/8)\right)
\end{aligned}$$

Example (4.2.11) Find the general solution of the differential equation $y'''-y''-y'+y=0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y=e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y=e^{rt}, y'=re^{rt}, y''=r^2e^{rt}, y'''=r^3e^{rt}.$$

$$\begin{aligned}
y'''-y''-y'+y &= 0\\
r^3e^{rt}-r^2e^{rt}-re^{rt}+e^{rt} &= 0\\
r^3-r^2-r+1 &= 0
\end{aligned}$$

Now we need the roots of the characteristic equation. We can guess one root, and then factor it out and use the quadratic formula, or use *Mathematica* to get all the roots at once. We find the roots to be $r_1=-1$, and $r_2=1$ of multiplicity 2, that is the characteristic equation can be written as

$$(r-1)^2(r+1)=0.$$

Therefore, a general solution of the differential equation is $y(t)=c_1e^{-t}+c_2e^t+c_3te^t$.

Example (4.2.13) Find the general solution of the differential equation $2y'''-4y''-2y'+4y=0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y=e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y=e^{rt}, y'=re^{rt}, y''=r^2e^{rt}, y'''=r^3e^{rt}.$$

$$\begin{aligned}
2y'''-4y''-2y'+4y &= 0\\
2r^3e^{rt}-4r^2e^{rt}-2re^{rt}+4e^{rt} &= 0\\
2r^3-4r^2-2r+4 &= 0
\end{aligned}$$

Now we need the roots of the characteristic equation. We can guess one root, and then factor it out and use the quadratic formula, or use *Mathematica* to get all the roots at once. We find the roots to be $r_1=1$, and $r_2=-1$, and $r_3=2$ of multiplicity 1, that is the characteristic equation can be written as

$$(r-1)(r+1)(r-2)=0.$$

Therefore, a general solution of the differential equation is $y(t) = c_1e^{-t} + c_2e^{-t} + c_3e^{2t}$.

Example (4.2.18) Find the general solution of the differential equation $y^{(6)} - y'' = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y^{(i)} = r^ie^{rt}.$$

$$\begin{aligned}
y^{(6)} - y'' &= 0 \\
r^6y^{(6)} - r^2e^{rt} &= 0 \\
r^2(r^4 - 1) &= 0 \\
r^2(r^2 - 1)(r^2 + 1) &= 0 \quad \text{difference of squares} \\
r^2(r - 1)(r + 1)(r - i)(r + i) &= 0
\end{aligned}$$

Therefore, $r_1 = 1$, and $r_2 = -1$, and $r_3 = i$, and $r_4 = -i$ of multiplicity one, and $r_5 = 0$ of multiplicity two. Therefore, a general solution of the differential equation is $y(t) = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t + c_5 + c_6t$.