Section 4.3: Nonhomogeneous Equations with Constant Coefficients: Undetermined Coefficients

Example (4.3.1) Find the general solution of the differential equation $y''' - y'' - y' + y = 2e^{-t} + 3$.

First, solve the associated homogeneous differential equation: $y''' - y'' - y' + y = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}, y''' = r^3e^{rt}.$$

$$\begin{aligned} y''' - y'' - y' + y &= 0 \\ r^3e^{rt} - r^2e^{rt} - re^{rt} + e^{rt} &= 0 \\ r^3 - r^2 - r + 1 &= 0 \end{aligned}$$

Now we need the roots of the characteristic equation. We can guess one root, and then factor it out and use the quadratic formula, or use *Mathematica* to get all the roots at once. We find the roots to be $r_1 = -1$, and $r_2 = 1$ of multiplicity 2, that is the characteristic equation can be written as

$$(r-1)^2(r+1) = 0.$$

Therefore, the complimentary solution is $y_c(t) = c_1e^{-t} + c_2e^t + c_3te^t$.

For undetermined coefficients, we assume a solution has the form $Y(t) = Ate^{-t} + B$. The first term was multiplied by t since Ae^{-t} appeared in the complimentary solution $y_c(t)$.

Take the derivatives and substitute into the nonhomogeneous differential equation. Although straightforward, this process can be tedious and you may prefer to have a computer algebra system assist you with the details. I will work it through by hand here.

$$\begin{aligned} Y(t) &= Ate^{-t} + B \\ Y'(t) &= Ae^{-t} - Ate^{-t} \\ Y''(t) &= -2Ae^{-t} + Ate^{-t} \\ Y'''(t) &= 3Ae^{-t} - Ate^{-t} \\ y''' - y'' - y' + y &= 2e^{-t} + 3 \\ (3Ae^{-t} - Ate^{-t}) - (-2Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) + (Ate^{-t} + B) &= 2e^{-t} + 3 \\ 4Ae^{-t} + B &= 2e^{-t} + 3 \end{aligned}$$

So we compare coefficients, and for this to be true for all values of t, we need $4A = 2$ and $B = 3$. So $A = 1/2$, $B = 3$.

The general solution to the nonhomogeneous differential equation is therefore $y(t) = c_1e^{-t} + c_2e^t + c_3te^t + \frac{t}{2}e^{-t} + 3$.

Example (4.3.4) Find the general solution of the differential equation $y''' - y' = 2\sin t$.

First, solve the associated homogeneous differential equation: $y''' - y' = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}, y''' = r^3e^{rt}.$$

$$\begin{aligned} y''' - y' &= 0 \\ r^3e^{rt} - re^{rt} &= 0 \\ r^3 - r &= 0 \\ r(r^2-1) &= 0 \\ r(r-1)(r+1) &= 0 \end{aligned}$$

We find the roots to be $r_1 = 0$, $r_2 = 1$, and $r_3 = -1$. Therefore, the complimentary solution is $y_c(t) = c_1 + c_2e^t + c_3e^{-t}$.

For undetermined coefficients, we assume a solution has the form $Y(t) = A\sin t + B\cos t$.

Take the derivatives and substitute into the nonhomogeneous differential equation.

$$\begin{aligned} Y(t) &= A\sin t + B\cos t \\ Y'(t) &= A\cos t - B\sin t \\ Y''(t) &= -A\sin t - B\cos t \\ Y'''(t) &= -A\cos t + B\sin t \\ y''' - y' &= 2\sin t \\ (-A\cos t + B\sin t) - (A\cos t - B\sin t) &= 2\sin t \\ 2B\sin t - 2A\cos t &= 2\sin t \end{aligned}$$

So we compare coefficients, and for this to be true for all values of t, we need $2B = 2$ and $2A = 0$. So $A = 0$, $B = 1$.

The general solution to the nonhomogeneous differential equation is therefore $y(t) = c_1 + c_2e^t + c_3e^{-t} + \cos t$.

Section 4.4: Nonhomogeneous Equations with Constant Coefficients: Variation of Parameters

Example (4.4.1) Find the general solution of the differential equation $y''' + y' = \tan t$.

First, solve the associated homogeneous differential equation: $y''' + y' = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}, y''' = r^3e^{rt}.$$

$$\begin{aligned} y''' + y' &= 0 \\ r^3 + r &= 0 \\ r(r+i)(r-i) &= 0 \end{aligned}$$

$r_1 = 0$, $r_2 = -i$, and $r_3 = +i$. We can group the last two solutions together as $r_{2,3} = \pm i = \lambda \pm \mu i$, $\lambda = 0$, $\mu = 1$. Two real valued solutions associated with these roots are $y_2 = e^{\lambda t}\cos\mu t = \cos t$, and $y_3 = e^{\lambda t}\sin\mu t = \sin t$.

Therefore, the complimentary solution is $y_c(t) = c_1 + c_2\cos t + c_3\sin t$.

Using variation of parameters (undetermined coefficients will not work in this case), we assume a solution of the nonhomogeneous differential equation looks like

$$Y(t) = \mu_1(t) + \mu_2(t)\cos t + \mu_3(t)\sin t = \mu_1 + \mu_2\cos t + \mu_3\sin t.$$

Take the derivative:

$$Y'(t) = \mu_1' + \mu_2' \cos t + \mu_3' \sin t - \mu_2 \sin t + \mu_3 \cos t.$$

Assume (this is a condition we impose on the problem):

$$\mu_1' + \mu_2' \cos t + \mu_3' \sin t = 0 \tag{1}$$

Therefore,

$$Y'(t) = -\mu_2 \sin t + \mu_3 \cos t.$$

Differentiate:

$$Y''(t) = -\mu_2' \sin t + \mu_3' \cos t - \mu_2 \cos t - \mu_3 \sin t.$$

Assume (this is the second condition we impose on the problem):

$$-\mu_2' \sin t + \mu_3' \cos t = 0 \tag{2}$$

Therefore,

$$Y''(t) = -\mu_2 \cos t - \mu_3 \sin t.$$

Differentiate:

$$Y'''(t) = -\mu_2' \cos t - \mu_3' \sin t + \mu_2 \sin t - \mu_3 \cos t.$$

Substitute into the differential equation, and simplify:

$$\begin{aligned} y''' + y' &= \tan t \\ (-\mu_2' \cos t - \mu_3' \sin t + \mu_2 \sin t - \mu_3 \cos t) + (-\mu_2 \sin t + \mu_3 \cos t) &= \tan t \\ -\mu_2' \cos t - \mu_3' \sin t &= \tan t \end{aligned} \tag{3}$$

If the μ_i do not cancel out at this stage, you have made an error!

Equations (1)–(3) form three equations in the three unknowns μ_1', μ_2', μ_3'. There are rewritten here so we can use Cramer's rule to solve the system:

$$\begin{array}{rcrcrcr} \mu_1' & + & \mu_2' \cos t & + & \mu_3' \sin t & = & 0 \\ & - & \mu_2' \sin t & + & \mu_3' \cos t & = & 0 \\ & - & \mu_2' \cos t & - & \mu_3' \sin t & = & \tan t \end{array}$$

$$\mu_1' = \frac{\begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \tan t & -\cos t & -\sin t \end{vmatrix}}{\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix}} = \frac{\tan t}{1} = \tan t$$

$$\mu_1 = \int \tan t \, dt = -\ln|\cos t| = \ln|\sec t|.$$

$$\mu_2' = \frac{\begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \tan t & -\sin t \end{vmatrix}}{1} = -\tan t \cos t$$

$$\mu_2 = -\int \tan t \cos t \, dt = -\int \sin t \, dt = \cos t.$$

$$\mu_3' = \frac{\begin{vmatrix} 1 & 0 & \cos t \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \tan t \end{vmatrix}}{1} = -\tan t \sin t$$

$$\mu_2 = -\int \tan t \sin t \, dt = -2\tanh^{-1}(\tan(t/2)) + \sin t.$$

This last integral was performed using *Mathematica*.

A particular solution to the nonhomogeneous differential equation is therefore

$$\begin{aligned} y_p(t) &= \mu_1 + \mu_2 \cos t + \mu_3 \sin t \\ &= \ln|\sec t| + \cos^2 t + \left(-2\tanh^{-1}(\tan(t/2)) + \sin t\right)\sin t \\ &= 1 + \ln|\sec t| - 2\tanh^{-1}(\tan(t/2))\sin t \end{aligned}$$

A general solution to the nonhomogeneous differential equation is therefore

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1 + c_2 \cos t + c_3 \sin t + 1 + \ln|\sec t| - 2\tanh^{-1}(\tan(t/2))\sin t \\ &= \tilde{c}_1 + c_2 \cos t + c_3 \sin t + \ln|\sec t| - 2\tanh^{-1}(\tan(t/2))\sin t \end{aligned}$$

where $\tilde{c}_1 = c_1 + 1$.

This solution can be quickly verified using *Mathematica*:

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DSolve[y'''[t] + y'[t] == Tan[t], y[t], t]
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If you have a few minutes, try to solve $y''' - y' = \tan t$ using *Mathematica*. I do not recommend doing this one by hand! Although all that was changed was a minus sign, the solution is much more complicated. It is because the integrals to determine $\mu_i(t)$ become quite involved.

Example (4.4.3) Find the general solution of the differential equation $y''' - 2y'' - y' + 2y = e^{4t}$.

First, solve the associated homogeneous differential equation: $y''' - 2y'' - y' + 2y = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y' = re^{rt}, y'' = r^2 e^{rt}, y''' = r^3 e^{rt}.$$

$$\begin{aligned} y''' - 2y'' - y' + 2y &= 0 \\ r^3 - 2r^2 - r + 2r &= 0 \end{aligned}$$

The roots can be found using *Mathematica*, or you can factor out $r - 1$, since $r = 1$ is a root by inspection. We find the roots to be $r_1 = 1$, $r_2 = -1$, and $r_3 = 2$.

Therefore, the complimentary solution is $y_c(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}$.

Using variation of parameters (undetermined coefficients will also work, and probably be much easier!), we assume a solution of the nonhomogeneous differential equation looks like

$$Y(t) = \mu_1(t)e^t + \mu_2(t)e^{-t} + \mu_3(t)e^{2t} = \mu_1 e^t + \mu_2 e^{-t} + \mu_3 e^{2t}.$$

Take the derivative:

$$Y'(t) = \mu_1' e^t + \mu_2' e^{-t} + \mu_3' e^{2t} + \mu_1 e^t - \mu_2 e^{-t} + 2\mu_3 e^{2t}.$$

Assume (this is a condition we impose on the problem):

$$\mu_1' e^t + \mu_2' e^{-t} + \mu_3' e^{2t} = 0 \tag{4}$$

Therefore,

$$Y'(t) = \mu_1 e^t - \mu_2 e^{-t} + 2\mu_3 e^{2t}.$$

Differentiate:

$$Y''(t) = \mu_1' e^t - \mu_2' e^{-t} + 2\mu_3' e^{2t} + \mu_1 e^t + \mu_2 e^{-t} + 4\mu_3 e^{2t}.$$

Assume (this is the second condition we impose on the problem):

$$\mu_1' e^t - \mu_2' e^{-t} + 2\mu_3' e^{2t} = 0 \tag{5}$$

Therefore,

$$Y''(t) = \mu_1 e^t + \mu_2 e^{-t} + 4\mu_3 e^{2t}.$$

Differentiate:

$$Y'''(t) = \mu_1' e^t + \mu_2' e^{-t} + 4\mu_3' e^{2t} + \mu_1 e^t - \mu_2 e^{-t} + 8\mu_3 e^{2t}.$$

Substitute into the differential equation, and simplify (details left out this time):

$$\mu_1' e^t + \mu_2' e^{-t} + 4\mu_3' e^{2t} \quad = \quad e^{4t} \tag{6}$$

If the μ_i do not cancel out at this stage, you have made an error!

Equations (4)–(6) form three equations in the three unknowns μ_1', μ_2', μ_3'. There are rewritten here so we can use Cramer's rule to solve the system:

$$\begin{array}{ccccccc} \mu_1' e^t & + & \mu_2' e^{-t} & + & \mu_3' e^{2t} & = & 0 \\ \mu_1' e^t & - & \mu_2' e^{-t} & + & 2\mu_3' e^{2t} & = & 0 \\ \mu_1' e^t & + & \mu_2' e^{-t} & + & 4\mu_3' e^{2t} & = & e^{4t} \end{array}$$

$$\mu_1' = \frac{\begin{vmatrix} 0 & e^{-t} & e^{2t} \\ 0 & -e^{-t} & 2e^{2t} \\ e^{4t} & e^{-t} & 4e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{t} & e^{-t} & e^{2t} \\ e^{t} & -e^{-t} & 2e^{2t} \\ e^{t} & e^{-t} & 4e^{2t} \end{vmatrix}} = \frac{e^{4t}(2e^{t}+e^{t})}{-6e^{2t}} = -\frac{1}{2}e^{3t}$$

$$\mu_1 = -\int \frac{1}{2}e^{3t}\,dt = -\frac{1}{6}e^{3t}.$$

$$\mu_2' = \frac{\begin{vmatrix} e^{t} & 0 & e^{2t} \\ e^{t} & 0 & 2e^{2t} \\ e^{t} & e^{4t} & 4e^{2t} \end{vmatrix}}{-6e^{2t}} = \frac{-e^{4t}(2e^{3t}-e^{3t})}{-6e^{2t}} = \frac{1}{6}e^{5t}$$

$$\mu_2 = \int \frac{1}{6}e^{5t}\,dt = \frac{1}{30}e^{5t}.$$

$$\mu_3' = \frac{\begin{vmatrix} e^{t} & e^{-t} & 0 \\ e^{t} & -e^{-t} & 0 \\ e^{t} & e^{-t} & e^{4t} \end{vmatrix}}{-6e^{2t}} = \frac{e^{4t}(-1-1)}{-6e^{2t}} = \frac{1}{3}e^{2t}$$

$$\mu_2 = \int \frac{1}{3}e^{2t}\,dt = \frac{1}{6}e^{2t}.$$

A particular solution to the nonhomogeneous differential equation is therefore

$$\begin{aligned} y_p(t) &= \mu_1 e^{t} + \mu_2 e^{-t} + \mu_3 e^{2t} \\ &= -\frac{1}{6}e^{3t}\cdot e^{t} + \frac{1}{30}e^{5t}\cdot e^{-t} + \frac{1}{6}e^{2t}\cdot e^{2t} \\ &= \frac{1}{30}e^{4t} \end{aligned}$$

A general solution to the nonhomogeneous differential equation is therefore

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + \frac{1}{30}e^{4t} \end{aligned}$$