Section 5.1: Review of Power Series

Example (5.1.1) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty}(x-3)^n$.

Use the ratio test: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ converges.

Here, $a_n = (x-3)^n$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| \\ &= \lim_{n\to\infty}|x-3| \\ &= |x-3|\end{aligned}$$

So we require this to be less than one for convergence, which means $|x-3| < 1$. Comparing this to the form $|x-x_0| < \rho$ tells us that this series is centered at $x_0 = 3$ and has radius of convergence of $\rho = 1$.

The interval of convergence is $-1 < x-3 < 1 \longrightarrow 2 < x < 4$. We would have to check the endpoints separately to find out if the series converges at $x = 2$ or $x = 4$.

Example (5.1.8) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{n!x^n}{n^n}$.

Use the root test: if $\lim_{n\to\infty}|a_n|^{1/n} < 1$ then $\sum a_n$ converges.

Here, $a_n = (n!x^n)/n^n$.

$$\begin{aligned}\lim_{n\to\infty}|a_n|^{1/n} &= \lim_{n\to\infty}\left|\frac{n!x^n}{n^n}\right|^{1/n} \\ &= \lim_{n\to\infty}\left|\frac{n!^{1/n}x}{n}\right|\end{aligned}$$

Hmmm, don't quite know what to do next.

Try the ratio test instead: if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum a_n$ converges.

Here, $a_n = (n!x^n)/n^n$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!x^n}\right| \\ &= \lim_{n\to\infty}\left|x\left(\frac{n}{n+1}\right)^n\right| \\ &= |x|\lim_{n\to\infty}\left|\left(1-\frac{1}{n+1}\right)^n\right|\end{aligned}$$

OK, let's play a bit. Or, fire up a computer to work out the limit.

$$\begin{aligned}\lim_{n\to\infty}\left|\left(1-\frac{1}{n+1}\right)^n\right| &= \lim_{m\to\infty}\left|\left(1-\frac{1}{m}\right)^{m-1}\right| \\ &= \lim_{m\to\infty}\left|\frac{(1-\frac{1}{m})^m}{(1-\frac{1}{m})}\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{\lim_{m\to\infty} \left(1 - \frac{1}{m}\right)^m}{\lim_{m\to\infty} \left(1 - \frac{1}{m}\right)} \right| \\
&= \left| \frac{\lim_{m\to\infty} \left(1 - \frac{1}{m}\right)^m}{(1 - 0)} \right| \\
&= \lim_{m\to\infty} \left| \left(1 - \frac{1}{m}\right)^m \right| \\
&= \lim_{n\to\infty} \left| \left(1 - \frac{1}{n}\right)^n \right| = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n
\end{aligned}
$$

Recall from Calculus the following result:

We can express e^α as the limit of a function. Here's how:

$$
\begin{aligned}
f(x) &= \ln x \\
f'(x) &= \frac{1}{x} \\
f'(1) &= 1
\end{aligned}
$$

Now, let's calculate $f'(1)$ using the definition of derivative.

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{x\to 0} \frac{f(1+x) - f(1)}{x} \text{ (relabel } h \to x) \\
&= \lim_{x\to 0} \frac{\ln(1+x) - \ln(1)}{x}, \quad \ln 1 = 0 \\
&= \lim_{x\to 0} \frac{1}{x} \ln(1+x) \text{ (use logarithm law } r \ln x = \ln(x^r)) \\
&= \lim_{x\to 0} \ln(1+x)^{1/x} \\
&= \ln \lim_{x\to 0} (1+x)^{1/x} \text{ (interchange limit and logarithm)}
\end{aligned}
$$

We know this should equal 1 from our above calculation. We also know that $\ln e = 1$. Therefore, we must have:

$$
e = \lim_{x\to 0} (1+x)^{1/x} \text{ (limit at zero)}
$$

Therefore,

$$
\begin{aligned}
e^\alpha &= \left(\lim_{x\to 0} (1+x)^{1/x} \right)^\alpha \\
&= \lim_{x\to 0} (1+x)^{\alpha/x} \quad \text{let } n = \alpha/x. \text{ As } x \longrightarrow 0, n \longrightarrow \infty \\
&= \lim_{n\to\infty} \left(1 + \frac{\alpha}{n}\right)^n \\
e^{-1} &= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n
\end{aligned}
$$

So, collecting this all together, we have:

$$
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| e^{-1}
$$

So we require this to be less than one for convergence, which means $|x|e^{-1} < 1$, or $|x| < e$. Comparing this to the form $|x - x_0| < \rho$ tells us that this series is centered at $x_0 = 0$ and has radius of convergence of $\rho = e$.

The interval of convergence is $-e < x < e$. We would have to check the endpoints separately to find out if the series converges at $x = \pm e$.

Example (5.1.15) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 0$ for the function $f(x) = \dfrac{1}{1-x}$.

$$f(x) = \frac{1}{1-x}, \quad x_0 = 0.$$

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$(1-x)^{-1}$	$1 = 0!$
1	$1(1-x)^{-2}$	$1 = 1!$
2	$2(1-x)^{-3}$	$1 \cdot 2 = 2!$
3	$2 \cdot 3(1-x)^{-4}$	$1 \cdot 2 \cdot 3 = 3!$
4	$2 \cdot 3 \cdot 4(1-x)^{-5}$	$1 \cdot 2 \cdot 3 \cdot 4 = 4!$
5	$2 \cdot 3 \cdot 4 \cdot 5(1-x)^{-6}$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5!$
	$\vdots$	
n	$n!(1-x)^{-(n+1)}$	$n!$

The Taylor series is $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{\infty} x^n$.

Radius of convergence: Use ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x| < 1$$

So radius of convergence is $\rho = 1$.

Example (5.1.16) Determine the Taylor Series (which includes the radius of convergence) about $x_0 = 2$ for the function $f(x) = \dfrac{1}{1-x}$.

$$f(x) = \frac{1}{1-x}, \quad x_0 = 2.$$

n	$f^{(n)}(x)$	$f^{(n)}(x_0)$
0	$(1-x)^{-1}$	$1 = 0!$
1	$1(1-x)^{-2}$	$1(-1)^{-2} = 1!$
2	$2(1-x)^{-3}$	$1 \cdot 2(-1)^{-3} = -2!$
3	$2 \cdot 3(1-x)^{-4}$	$1 \cdot 2 \cdot 3(-1)^{-4} = 3!$
4	$2 \cdot 3 \cdot 4(1-x)^{-5}$	$1 \cdot 2 \cdot 3 \cdot 4(-1)^{-5} = -4!$
5	$2 \cdot 3 \cdot 4 \cdot 5(1-x)^{-6}$	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5(-1)^{-6} = 5!$
	$\vdots$	
n	$n!(-1)^{n+1}(1-x)^{-(n+1)}$	$(-1)^{n+1}n!$

The Taylor series is $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n$.

Radius of convergence: Use ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\right| = \lim_{n\to\infty}|(x-2)| = |x-2| < 1$$

So radius of convergence is $\rho = 1$.

Example (5.1.21) Rewrite $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ into a sum whose generic term involves x^n.

Here, we want to let $m = n-2$. When $n = 2$, $m = 0$. Also, $n = m+2$.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$$

Now, relabel as $m = n$:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Remember, n is a dummy index and that is what makes this relabelling possible.

Example (5.1.25) Rewrite $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1}$ into a sum whose generic term involves x^n.

Here, we want to let $n = m-2$. When $m = 2$, $n = 0$. Also, $m = n+2$. Also, we can bring the x inside the summation in the second summation.

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} k a_k x^k$$

Now, relabel as $k = n$. Also, we need to take the $n = 0$ term out of the first sum, so both sums will begin at $n = 1$.

$$\begin{aligned}\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} k a_k x^k \\ &= 2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n \\ &= 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + n a_n\right]x^n\end{aligned}$$

In this case, we can collect the first term in with the rest, since the quantity in the square brackets is $2a_2$ when $n = 0$. This final simplification will not always occur when we are looking for series solutions of differential equations!

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + n a_n\right]x^n$$