Section 5.2: Series Solution Near Ordinary Point, Part I

Example (5.2.1) Solve the differential equation $y'' - y = 0$ using a series solution about $x_0 = 0$.

This could be solved by assuming $y = e^{rt}$, since the differential equation has constant coefficients and is linear. We will solve using series solution instead.

First, since $p(x) = 0$ and $q(x) = 1$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' - y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Relabel each term so it has an x^n:

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - a_n \right] x^n &= 0 \end{aligned}$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$(n+1)(n+2) a_{n+2} - a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

This is the recurrence relation. We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern. We will not always be able to determine a pattern!

$$\begin{aligned} a_{n+2} &= \frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, 3, \ldots \\ a_0 &= \text{unspecified, assume not equal to zero} \\ a_1 &= \text{unspecified, assume not equal to zero} \\ a_2 &= \frac{a_0}{2} = \frac{a_0}{2!} \\ a_3 &= \frac{a_1}{6} = \frac{a_1}{3!} \\ a_4 &= \frac{a_2}{12} = \frac{a_0}{4!} \\ a_5 &= \frac{a_3}{20} = \frac{a_1}{5!} \end{aligned}$$

The pattern in the above is $a_{2n} = a_0/(2n)!$ for even terms, and for odd terms we get $a_{2n+1} = a_1/(2n+1)!$. Therefore,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{a_0}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{a_1}{(2n+1)!} x^{2n+1} \\ &= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ &= a_0 \cosh x + a_1 \sinh x \end{aligned}$$

We were able to sum the infinite series, or more precisely we recognized them as Taylor series expansions of known functions. This is what we would really like to be able to do all the time, but it is not always possible.

The a_0 and a_1 are the constants of integration which would be determined by initial conditions if we had an initial value problem. We might prefer to write the solution as $y(x) = c_1 y_1(x) + c_2 y_2(x)$, where $y_1(x) = \cosh x$ and $y_2(x) = \sinh x$ form a fundamental set of solutions.

Example (5.2.2) Solve the differential equation $y'' - xy' - y = 0$ using a series solution about $x_0 = 0$.

This could not be solved by assuming $y = e^{rt}$, since the differential equation has variable coefficients.

First, since $p(x) = -x$ and $q(x) = -1$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' - xy' - y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Relabel each term so it has an x^n:

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now we need each sum to start at the same value of n; we can achieve this by removing the $n = 0$ terms from the first and third sum:

$$(2a_2 - a_0)x^0 + \sum_{n=1}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = 0$$

$$(2a_2 - a_0)x^0 + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} - na_n - a_n]\, x^n = 0$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$\begin{aligned} 2a_2 - a_0 &= 0 \\ (n+1)(n+2)a_{n+2} - na_n - a_n &= 0, \quad n = 1, 2, 3, \ldots \end{aligned}$$

Notice that since the sum started at $n = 1$, the second equation is true for $n = 1, 2, 3, \ldots$.

These are the recurrence relations. Sometimes (but not always!) it is the case that the recurrence relations can be written for $n = 0, 1, 2, 3, \ldots$. We see here that $n = 0$ in the second equation gives us $2a_2 - a_0 = 0$, so the first equation is really the second with $n = 0$. A bit of algebra gives us for the recurrence relations (since $n + 1 \neq 0$):

$$(n+2)a_{n+2} - a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern.

$$\begin{aligned} a_{n+2} &= \frac{a_n}{(n+2)}, \quad n = 0, 1, 2, 3, \ldots \\ a_0 &= c_1 \text{ unspecified, assume not equal to zero} \\ a_1 &= c_2 \text{ unspecified, assume not equal to zero} \\ a_2 &= \frac{a_0}{2} = \frac{c_1}{2} \\ a_3 &= \frac{a_1}{3} = \frac{c_2}{3} \\ a_4 &= \frac{a_2}{4} = \frac{c_1}{2 \cdot 4} \\ a_5 &= \frac{a_3}{5} = \frac{c_2}{3 \cdot 5} \end{aligned}$$

The pattern in the above is $a_{2k} = c_1/(2 \cdot 4 \cdot 6 \cdots (2k))$ for even terms, and for odd terms we get $a_{2k+1} = c_2/(1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1))$. Therefore,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1} \\ &= c_1 \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)} + c_2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)} \\ &= c_1 y_1(x) + c_2 y_2(x) \\ y_1(x) &= \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)} \\ y_2(x) &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)} \end{aligned}$$

The $y_1(x)$ and $y_2(x)$ are linearly independent since one is odd and the other even, so they form a fundamental set of solutions.

We would like to simplify these functions if possible, so let's see what we can do!

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdots (2n)} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n (1 \cdot 2 \cdot 3 \cdots n)} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2} \right)^n \\ &= e^{x^2/2} \end{aligned}$$

where we recognized the Taylor series $e^u = \displaystyle\sum_{n=0}^{\infty} \frac{u^n}{n!}$.

The second function is trickier. Notice that we almost have a factorial in the denominator, but we are missing $2 \cdot 4 \cdot 6 \cdots (2n) = 2(1 \cdot 2 \cdot 3 \cdots n)$:

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1} 2(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{2n! x^{2n+1}}{(2n+1)!} \end{aligned}$$

This is an improvement, since we have removed the $\cdots$ and now have factorials. We can leave it here, but if you fire up *Mathematica* you can simplify this even further.

$$y_2(x) = \sum_{n=0}^{\infty} \frac{2n! x^{2n+1}}{(2n+1)!} = 2e^{x^2/2} \sqrt{\pi}\, \operatorname{erf}(x/2).$$

The error function $\operatorname{erf}(x)$ is given by $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$, and it is commonly seen in probability theory.

Example (5.2.21) Hermite Equation Solve the differential equation $y'' - 2xy' + \lambda y = 0$ using a series solution about $x_0 = 0$.

This could <u>not</u> be solved by assuming $y = e^{rt}$, since the differential equation has variable coefficients.

First, since $p(x) = -2x$ and $q(x) = \lambda$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
y'' - 2xy' + \lambda y &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n &= 0
\end{aligned}$$

Relabel each term so it has an x^n:

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n = 0$$

Now we need each sum to start at the same value of n; we can achieve this by removing the $n = 0$ terms from the first and third sum:

$$\begin{aligned}
(2a_2 + \lambda a_0)x^0 + \sum_{n=1}^{\infty} (n+1)(n+2)a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=1}^{\infty} \lambda a_n x^n &= 0 \\
(2a_2 + \lambda a_0)x^0 + \sum_{n=1}^{\infty} \left[(n+1)(n+2)a_{n+2} - 2n a_n + \lambda a_n\right] x^n &= 0
\end{aligned}$$

For this to be true for all values of x, each coefficient of the series must be zero,

$$\begin{aligned}
2a_2 + \lambda a_0 &= 0 \\
(n+1)(n+2)a_{n+2} - 2n a_n + \lambda a_n &= 0, \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Notice that since the sum started at $n = 1$, the second equation is true for $n = 1, 2, 3, \ldots$.

These are the recurrence relations. Sometimes (but not always!) it is the case that the recurrence relations can be written for $n = 0, 1, 2, 3, \ldots$. We see here that $n = 0$ in the second equation gives us $2a_2 - \lambda a_0 = 0$, so the first equation is really the second with $n = 0$. A bit of algebra gives us for the recurrence relations :

$$(n+1)(n+2)a_{n+2} + (\lambda - 2n)a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

We solve the recurrence relation for a_{n+2}, then determine the first few coefficients a_i and try to determine a pattern.

$$\begin{aligned}
a_{n+2} &= \frac{a_n(2n - \lambda)}{(n+1)(n+2)}, \quad n = 0, 1, 2, 3, \ldots \\
a_0 &= c_1 \text{ unspecified, assume not equal to zero} \\
a_1 &= c_2 \text{ unspecified, assume not equal to zero} \\
a_2 &= \frac{a_0(-\lambda)}{2} = \frac{c_1(-\lambda)}{2} \\
a_3 &= \frac{a_1(2 - \lambda)}{2 \cdot 3} = \frac{c_2(2 - \lambda)}{3!} \\
a_4 &= \frac{a_2(4 - \lambda)}{3 \cdot 4} = \frac{c_1(-\lambda)(4 - \lambda)}{4!} \\
a_5 &= \frac{a_3(6 - \lambda)}{4 \cdot 5} = \frac{c_2(2 - \lambda)(6 - \lambda)}{5!} \\
a_6 &= \frac{a_4(8 - \lambda)}{5 \cdot 6} = \frac{c_1(-\lambda)(4 - \lambda)(8 - \lambda)}{6!} \\
a_7 &= \frac{a_5(10 - \lambda)}{6 \cdot 7} = \frac{c_2(2 - \lambda)(6 - \lambda)(10 - \lambda)}{7!}
\end{aligned}$$

The pattern in the above is a bit difficult to write out, but it is readily apparent there is a pattern! For even terms, the pattern is

$$\begin{aligned}
a_{2k+2} &= c_1 \frac{(2\cdot 0-\lambda)(2\cdot 2-\lambda)(2\cdot 4-\lambda)(2\cdot 6-\lambda)\cdots(2(2k)-\lambda)}{(2k+2)!},\ k=0,1,2,3,\ldots \\
&= c_1 \frac{(4\cdot 0-\lambda)(4\cdot 1-\lambda)(4\cdot 2-\lambda)(4\cdot 3-\lambda)\cdots(4k-\lambda)}{(2k+2)!},\ k=0,1,2,3,\ldots \\
&= c_1 \frac{1}{(2k+2)!}\prod_{i=0}^{k}(4\cdot i-\lambda), k=0,1,2,3,\ldots
\end{aligned}$$

For odd terms, the pattern is identified in terms of products using $\dfrac{(2\cdot 2-\lambda)(2\cdot 4-\lambda)(2\cdot 6-\lambda)\cdots(2\cdot 2k-\lambda)}{\prod_{i=1}^{k}(4\cdot i-\lambda)}=1$,

$$\begin{aligned}
a_{2k+3} &= c_2 \frac{(2\cdot 1-\lambda)(2\cdot 3-\lambda)(2\cdot 5-\lambda)\cdots(2(2k+1)-\lambda)}{(2k+3)!},\ k=0,1,2,3,\ldots \\
&= c_2 \frac{(2\cdot 1-\lambda)(2\cdot 2-\lambda)(2\cdot 3-\lambda)(2\cdot 4-\lambda)(2\cdot 5-\lambda)\cdots(2\cdot 2k-\lambda)(2(2k+1)-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)},\ k=0,1,2,3,\ldots \\
&= c_2 \frac{\prod_{i=1}^{2k+1}(2\cdot i-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)},\ k=0,1,2,3,\ldots
\end{aligned}$$

I have checked these patterns in the associated *Mathematica* file. That's always a good idea when you are doing some complicated simplifications!

Therefore,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{k=0}^{\infty} a_{2k+2}x^{2k+2} + a_1 x + \sum_{k=0}^{\infty} a_{2k+3}x^{2k+3} \\
&= c_1\left(1+\sum_{k=0}^{\infty}\frac{1}{(2k+2)!}\prod_{i=0}^{k}(4\cdot i-\lambda)x^{2k+2}\right) + c_2\left(x+\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{2k+1}(2\cdot i-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)}x^{2k+3}\right) \\
&= c_1 y_1(x) + c_2 y_2(x) \\
y_1(x) &= 1+\sum_{k=0}^{\infty}\frac{1}{(2k+2)!}\prod_{i=0}^{k}(4\cdot i-\lambda)x^{2k+2} \\
y_2(x) &= x+\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{2k+1}(2\cdot i-\lambda)}{(2k+3)!\prod_{i=1}^{k}(4\cdot i-\lambda)}x^{2k+3}
\end{aligned}$$

The $y_1(x)$ and $y_2(x)$ are linearly independent since one is odd and the other even, so they form a fundamental set of solutions. If we can't recognize the pattern, which we saw was a difficult process, we can instead write the first few terms in the series (usually four or five will do). If more terms are required, the coefficients can be calculated using the recursion relations.

The Hermite Polynomials

What follows is particularly of interest to physicists, since the Hermite polynomials $H_n(x)$ arise in solving the Schrödinger equation for a harmonic oscillator. However, it also shows one way in which special functions arise from differential equations, so in that sense it is of interest to all.

If λ is nonnegative even integer, then $\lambda = 2m$, and something interesting happens to our solutions. One of these solutions will become a polynomial in this case–the first if m is even, and the second if m is odd. Let's see how this happens.

Assume m is even.

$$\begin{aligned} y_1(x) &= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - \lambda) x^{2k+2} \\ &= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \prod_{i=0}^{k} (4 \cdot i - 2m) x^{2k+2} \\ &= 1 + \sum_{k=0}^{\infty} \frac{2^{k+1}}{(2k+2)!} \prod_{i=0}^{k} (2 \cdot i - m) x^{2k+2} \\ &= 1 + \sum_{k=0}^{m/2} \frac{2^{k+1}}{(2k+2)!} \prod_{i=0}^{k} (2 \cdot i - m) x^{2k+2} \end{aligned}$$

where we have stopped summing at $k = m/2$ (which is an integer since m is even) since higher terms will have a factor $2 \cdot m/2 - m = 0$ in the product.

Assume m is odd.

$$\begin{aligned} y_2(x) &= x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1}(2 \cdot i - \lambda)}{(2k+3)! \prod_{i=1}^{k}(4 \cdot i - \lambda)} x^{2k+3} \\ &= x + \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{2k+1}(2 \cdot i - 2m)}{(2k+3)! \prod_{i=1}^{k}(4 \cdot i - 2m)} x^{2k+3} \\ &= x + \sum_{k=0}^{(m-1)/2} \frac{2^{k+1}}{(2k+3)!} \frac{\prod_{i=1}^{2k+1}(\cdot i - m)}{\prod_{i=1}^{k}(2 \cdot i - m)} x^{2k+3} \end{aligned}$$

The product in the numerator will have a zero factor when $2k + 1 - m = 0$. Therefore, we stopped the summing at $k = (m-1)/2$. This is an integer since m is odd.

The Hermite polynomial $H_m(x)$ is defined as the polynomial solution to the Hermite equation with $\lambda = 2m$ for which the coefficient of x^m is 2^m. The Hermite polynomials are found from flipping back and forth between y_1 and y_2, depending on which one has the terminating infinite sum, and then normalizing.

m	$H_m(x)$	$y_1(x)\vert_m$	$y_2(x)\vert_m$
0	1	1	–
1	$2x$	–	$x = \frac{1}{2}(2x)$
2	$-2 + 4x^2$	$1 - 2x^2 = -\frac{1}{2}(-2 + 4x^2)$	–
3	$-12x + 8x^3$	–	$x - \frac{2}{3}x^3 = -\frac{1}{12}(-12x + 8x^3)$
4	$12 - 48x^2 + 16x^4$	$1 - 4x^2 + \frac{4}{3}x^4 = \frac{1}{12}(12 - 48x^2 + 16x^4)$	–
5	$120x - 160x^3 + 32x^5$	–	$x - \frac{4}{3}x^3 + \frac{4}{15}x^5 = \frac{1}{120}(120x - 160x^3 + 32x^5)$

What this means is that the differential equation $y'' - 2xy' + 2ny = 0$, n an integer, has a solution $H_n(x)$, which is a polynomial, not an infinite series. The other solution is an infinite series, and can be represented by a Hypergeometric function.

In physics, this differential equation arises when solving the quantum mechanical harmonic oscillator. The solution which is an infinite series is not physical, since it leads to a quantum mechanical wavefunction which is infinite as $x \to \infty$.