Section 5.3: Series Solution Near Ordinary Point, Part II

Example (5.3.5) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 4$ for the differential equation $y'' + 4y' + 6xy = 0$.

The point $x_0 = 0$ is an ordinary point since

$$p(x) = 4, \qquad q(x) = 6x$$

are both analytic about $x_0 = 0$.

The radius of convergence of the series solution will be at least as large as the minimum of the radius of convergence of the series for $p(x) = 4$ and $q(x) = 6x$ about $x_0 = 0$.

Since $p(x)$ and $q(x)$ are already expanded in power series, and these series are not infinite, the radius of convergence for them is $\rho = \infty$.

Therefore, the series solution about $x_0 = 0$ must have a radius of convergence that is at least as large as $\rho = \infty$, which of course means it must be $\rho = \infty$.

A similar argument holds for $x_0 = 4$.

Example (5.3.7) Determine a lower bound on the radius of convergence for the series solution about $x_0 = 0$ and $x_0 = 2$ for the differential equation $(1 + x^3)y'' + 4xy' + y = 0$.

The point $x_0 = 0$ is an ordinary point since

$$p(x) = \frac{4x}{1+x^3}, \qquad q(x) = \frac{1}{1+x^3}$$

are both analytic about $x_0 = 0$.

Let's determine the radius of convergence of p and q without working out the Taylor series for them.

The complex poles of p and q all occur when $1 - x^3 = 0$, which means $x = 1^{1/3}$, which is the third root of unity (studied in Chapter 4). These roots are $x = -1, 1/2 + i\sqrt{3}/2, 1/2 - i\sqrt{3}/2$.

The distance from $x_0 = 0$ to the nearest complex pole is 1 *(diagram in Mathematica file)*.

$$\text{distance } = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

Therefore, the radius of convergence of the series for $p(x)$ and $q(x)$ is $\rho = 1$.

The minimum radius of convergence for the series solution about $x_0 = 0$ to the differential equation is $\rho = 1$.

The distance from $x_0 = 2$ to the nearest complex pole is $\sqrt{3}$ *(diagram in Mathematica file)*.

$$\text{distance } = \sqrt{\left(2 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}.$$

Therefore, the radius of convergence of the series for $p(x)$ and $q(x)$ is $\rho = \sqrt{3}$.

The minimum radius of convergence for the series solution about $x_0 = 2$ to the differential equation is $\rho = \sqrt{3}$.

Example (5.3.11) Find the first four nonzero terms in two linearly independent series solutions about the origin to the differential equation $y'' + (\sin x)y = 0$. What do you expect the radius of convergence to be?

First, since $\sin x$ has a series solution about $x_0 = 0$ which converges for all x, we expect our series solution to converge for all x, which means the radius of convergence for the series solution should be $\rho = \infty$.

We need to expand the sine function, if we hope to collect powers of x.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Since $p(x) = 0$ and $q(x) = \sin x$, which are analytic about $x = 0$, the point $x = 0$ is an ordinary point. Therefore, the assumed solution for the differential equation is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' + (\sin x) y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \left(\sum_{n=0}^{\infty} a_n x^n \right) &= 0 \end{aligned}$$

Since we want to collect powers of x to get a recurrence relation, we have two options here. We can multiply out the two infinite series, or truncate all of them. There is an example of working with the infinite series multiplied together on a different differential equation at http://cda.mrs.umn.edu/ mcquarrb/DigitalDE/Math/SSOP.nb.

Let's truncate them here, and see what happens. Assuming that we will get both solutions at once, if we want the first four nonzero terms in each we should go out to at least x^8. The manipulations that lead to the following equation are somewhat tedious, and I used *Mathematica* to perform them (they are in the associated *Mathematica* file).

Let's choose to truncate them all at $k = 12$ (I encourage to investigate this with the associated *Mathematica* file), and then we get the following:

$$\begin{aligned} & 2a_2 + (a_0 + 6a_3)x + (a_1 + 12a_4)x^2 + \left(-\frac{a_0}{6} + a_2 + 20a_5 \right) x^3 + \left(-\frac{a_1}{6} + a_3 + 30a_6 \right) x^4 \\ & + \left(\frac{a_0}{120} - \frac{a_2}{6} + a_4 + 42a_7 \right) x^5 + \left(\frac{a_1}{120} - \frac{a_3}{6} + a_5 + 56a_8 \right) x^6 + \left(-\frac{a_0}{5040} + \frac{a_2}{120} - \frac{a_4}{6} + a_6 + 72a_9 \right) x^7 \\ & + \left(-\frac{a_1}{5040} + \frac{a_3}{120} - \frac{a_5}{6} + a_7 + 90a_{10} \right) x^8 + \ldots = 0 \end{aligned}$$

Make sure you keep enough terms in the expansion so that you aren't missing anything in the above. Set each coefficient of x to zero, and solve recursively for as many coefficients as you need (we were asked to get four nonzero terms in each solution).

$$\begin{aligned} a_0 &= a_0 \text{ unspecified, arbitrary not equal to zero} \\ a_1 &= a_1 \text{ unspecified, arbitrary not equal to zero} \\ 2a_2 &= 0 \longrightarrow a_2 = 0 \end{aligned}$$

$$\begin{aligned}
a_0 + 6a_3 &= 0 \longrightarrow a_3 = -\frac{a_0}{6} \\
a_1 + 12a_4 &= 0 \longrightarrow a_4 = -\frac{a_1}{12} \\
-\frac{a_0}{6} + a_2 + 20a_5 &= 0 \longrightarrow a_5 = \frac{a_0}{120} \\
-\frac{a_1}{6} + a_3 + 30a_6 &= 0 \longrightarrow a_6 = \frac{a_1}{180} + \frac{a_0}{180} \\
\frac{a_0}{120} - \frac{a_2}{6} + a_4 + 42a_7 &= 0 \longrightarrow a_7 = -\frac{a_0}{5040} + \frac{a_1}{504}
\end{aligned}$$

We can actually stop here. Notice we only used the first six terms in our expansion.

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots \\
&= a_0 + a_1 x - \frac{a_0}{6}x^3 - \frac{a_1}{12}x^4 + \frac{a_0}{120}x^5 + \frac{a_1}{180}x^6 + \frac{a_0}{180}x^6 - \frac{a_0}{5040}x^7 + \frac{a_1}{504}x^7 + \ldots \\
&= a_0\left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} - \frac{x^7}{5040} + \ldots\right) + \left(x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \ldots\right) \\
&= a_0 y_1(x) + a_1 y_2(x)
\end{aligned}$$

The two linearly independent solutions are

$$\begin{aligned}
y_1(x) &= 1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} - \frac{x^7}{5040} + \ldots \\
y_2(x) &= x - \frac{x^4}{12} + \frac{x^6}{180} + \frac{x^7}{504} + \ldots
\end{aligned}$$