Section 5.4: Euler's Equation; Regular Singular Points

Example (5.4.1) Determine the solution to the differential equation $x^2y'' + 4xy' + 2y = 0$ that is valid in any interval not containing the singular point.

The is an Euler equation (since the power of x in the coefficient is the same as the derivative), and the singular point is $x = 0$.

Assume a solution looks like $y = x^r$.

Differentiate, and substitute into the differential equation:

$$y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}$$

$$\begin{aligned} x^2y'' + 4xy' + 2y &= 0 \\ r(r-1)x^r + 4rx^r + 2x^r &= 0 \\ r(r-1) + 4r + 2 &= 0 \end{aligned}$$

This is the indicial equation. Solving, we find $(r+2)(r+1) = 0$, so the roots are $r = -2$ and $r = -1$.

The general solution is therefore $y(x) = c_1x^{-1} + c_2x^{-2}$, which is valid for $x > 0$. This is also the solution if $x < 0$, so the general solution is $y(x) = \dfrac{c_1}{x} + \dfrac{c_2}{x^2}, \ x \neq 0$.

Example (5.4.2) Determine the solution to the differential equation $(x+1)^2y'' + 3(x+1)y' + \dfrac{3}{4}y = 0$ that is valid in any interval not containing the singular point.

The is an Euler equation (since the power of $x+1$ in the coefficient is the same as the derivative), and the singular point is $x = -1$.

Assume a solution looks like $y = (x+1)^r$.

Differentiate, and substitute into the differential equation:

$$y = (x+1)^r, \quad y' = r(x+1)^{r-1}, \quad y'' = r(r-1)(x+1)^{r-2}$$

$$\begin{aligned} (x+1)^2y'' + 3(x+1)y' + \frac{3}{4}y &= 0 \\ r(r-1)(x+1)^r + 3r(x+1)^r + \frac{3}{4}(x+1)^r &= 0 \\ r(r-1) + 3r + \frac{3}{4} &= 0 \end{aligned}$$

This is the indicial equation. Solving, we find $(r+3/2)(r+1/2) = 0$, so the roots are $r = -3/2$ and $r = -1/2$.

The general solution is therefore $y(x) = c_1(x+1)^{-3/2} + c_2(x+1)^{-1/2}$, which is valid for $x > 1$. For $x < 1$, the solution is $y(x) = c_1|x+1|^{-3/2} + c_2|x+1|^{-1/2}$. For $x \neq 1$, the solution is $y(x) = c_1|x+1|^{-3/2} + c_2|x+1|^{-1/2}$.

Example (5.4.9) Determine the solution to the differential equation $x^2y'' - 5xy' + 9y = 0$ that is valid in any interval not containing the singular point.

The is an Euler equation (since the power of x in the coefficient is the same as the derivative), and the singular point is $x = 0$.

Assume a solution looks like $y = x^r$.

Differentiate, and substitute into the differential equation:

$$y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}$$

$$x^2y'' - 5xy' + 9y = 0$$

$$\begin{aligned} r(r-1)x^r - 5rx^r + 9x^r &= 0 \\ r(r-1) - 5r + 9 &= 0 \end{aligned}$$

This is the indicial equation. Solving, we find $(r-3)^2 = 0$, so the root is $r = 3$ of multiplicity 2.

Therefore, one solution is $y_1(x) = x^3$.

If we forget the second solution looks like $y_2(x) = x^3 \ln|x|$ for a repeated root, we can always work it out using reduction of order.

Assume $y(x) = v(x)y_1(x) = vx^3$.

$$\begin{aligned} y' &= v'x^3 + 3vx^2 \\ y'' &= v''x^3 + 6v'x^2 + 6vx \end{aligned}$$

Substitute into the differential equation, and determine v:

$$\begin{aligned} x^2y'' - 5xy' + 9y &= 0 \\ x^2(v''x^3 + 6v'x^2 + 6vx) - 5x(v'x^3 + 3vx^2) + 9(vx^3) &= 0 \\ v''x^5 + 6v'x^4 + \cancel{6vx^3} - 5v'x^4 - \cancel{15vx^3} + \cancel{9vx^3} &= 0 \\ v''x + v' &= 0 \\ \frac{dv'}{dx}x + v' &= 0 \\ \int \frac{dv'}{v'} &= -\int \frac{dx}{x} \\ \ln|v'| + c_1 &= -\ln|x| \\ \ln|v'| &= \ln|x^{-1}| - c_1 \\ |v'| &= e^{-c_1}|x^{-1}| \\ v' &= \frac{c_2}{x}, \quad c_2 = e^{c_1} \\ \int dv &= \int \frac{c_2}{x}\,dx \\ v &= c_2 \ln|x| + c_3 \end{aligned}$$

Another solution is therefore $y(x) = v(x)y_1(x) = c_2x^3 \ln|x| + c_3x^3$. This is the general solution, and a fundamental set of solutions is $y_1(x) = x^3$ and $y_2(x) = x^3 \ln|x|$, which is valid for $x \neq 0$.

Example (5.4.17) Find all the singular points of the differential equation $xy'' + (1-x)y' + xy = 0$, and determine whether each one is regular or irregular.

The general form of a linear second order differential equation is $y'' + p(x)y' + q(x)y = 0$.

Identify $p(x) = \dfrac{1-x}{x}$ and $q(x) = \dfrac{x}{x} = 1$.

Since $p(x)$ is not analytic at $x = 0$ (meaning there is no Taylor series with nonzero radius of convergence about $x = 0$), $x = 0$ is a singular point.

Since $xp(x) = 1 - x$ and $x^2q(x) = x^2$ are both analytic at $x = 0$, we have $x = 0$ is a regular singular point.

Example (5.4.20) Find all the singular points of the differential equation $x^2(1-x^2)y'' + \dfrac{2}{x}y' + 4y = 0$, and determine whether each one is regular or irregular.

The general form of a linear second order differential equation is $y'' + p(x)y' + q(x)y = 0$.

Identify $p(x) = \dfrac{2}{x^3(1-x^2)}$ and $q(x) = \dfrac{4}{x^2(1-x^2)}$.

Since $p(x)$ is not analytic at $x = 0, 1, -1$ (meaning there is no Taylor series with nonzero radius of convergence about $x = 0, 1, -1$), $x = 0, 1, -1$ are singular points. Also, $q(x)$ is not analytic at these points.

We need to classify these points, taking each in turn.

$\underline{x = 0}$:

Consider $(x-0)p(x) = \dfrac{2}{x^2(1-x^2)}$ and $(x-0)^2q(x) = \dfrac{4}{1-x^2}$. Although $x^2q(x)$ is analytic at $x = 0$, since $xp(x)$ is not analytic at $x = 0$, the point $x = 0$ is an irregular singular point.

$\underline{x = 1}$:

Consider $(x-1)p(x) = -\dfrac{2}{x^3(1+x)}$ and $(x-1)^2q(x) = \dfrac{4(1-x)}{x^3(1+x)}$. Both these are analytic at $x = 1$, so $x = 1$ is a regular singular point.

$\underline{x = -1}$:

Consider $(x+1)p(x) = \dfrac{2}{x^3(1-x)}$ and $(x+1)^2q(x) = \dfrac{4(1+x)}{x^3(1-x)}$. Both these are analytic at $x = -1$, so $x = -1$ is a regular singular point.

Example (5.4.21) Find all the singular points of the differential equation $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$, and determine whether each one is regular or irregular.

The general form of a linear second order differential equation is $y'' + p(x)y' + q(x)y = 0$.

Identify $p(x) = \dfrac{x(1-x)}{(1-x^2)^2} = \dfrac{x}{(1-x)(1+x)^2}$ and $q(x) = \dfrac{(1+x)}{(1-x^2)^2} = \dfrac{1}{(1-x)^2(1+x)}$.

Since $p(x)$ is not analytic at $x = \pm 1$ (meaning there is no Taylor series with nonzero radius of convergence about $x = \pm 1$), $x = \pm 1$ are singular points. Also, $q(x)$ is not analytic at these points.

We need to classify these points, taking each in turn.

$\underline{x = -1}$:

Consider $(x+1)p(x) = \dfrac{x}{(1-x)(1+x)}$ and $(x+1)^2q(x) = \dfrac{(x+1)}{(1-x)^2}$. Since $(x+1)p(x)$ is not analytic at $x = -1$, $x = -1$ is an irregular singular point.

$\underline{x = 1}$:

Consider $(x-1)p(x) = \dfrac{x}{(1+x)^2}$ and $(x-1)^2q(x) = \dfrac{1}{1+x}$. Both these are analytic at $x = 1$, so $x = 1$ is a regular singular point.

Example (5.4.41) For the differential equation $2xy'' + 3y' + xy = 0$, show that $x = 0$ is a regular singular point. Show that there is only one nonzero solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. In general, for regular singular points x_0 there may be no solutions of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Identify $p(x) = \dfrac{3}{2x}$ and $q(x) = \dfrac{1}{2}$.

Since $p(x)$ is not analytic at $x = 0$, $x = 0$ is a singular point.

Consider $xp(x) = \dfrac{3}{2}$ and $x^2q(x) = \dfrac{x^2}{2}$. Since both of these are analytic at $x = 0$, $x = 0$ is a regular singular point.

Assume:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
y'' &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n
\end{aligned}
$$

Substitute into the differential equation:

$$
\begin{aligned}
2x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + x\sum_{n=0}^{\infty}a_nx^n &= 0 \\
2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}a_nx^{n+1} &= 0 \\
2\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}a_{n-1}x^n &= 0 \\
2\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + 3a_1x^0 + 3\sum_{n=1}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty}a_{n-1}x^n &= 0 \\
3a_1x^0 + \sum_{n=1}^{\infty}\Big[2(n+1)na_{n+1} + 3(n+1)a_{n+1} + a_{n-1}\Big]x^n &= 0
\end{aligned}
$$

For this to be true for all values of x, the coefficients of powers of x must be zero. This leads to the relations:

$$
\begin{aligned}
3a_1 &= 0 \\
2(n+1)na_{n+1} + 3(n+1)a_{n+1} + a_{n-1} &= 0, \ \ n = 1, 2, 3, \ldots
\end{aligned}
$$

These are the recurrence relations. Solving for the coefficients, we get

$$
\begin{aligned}
a_{n+1} &= \frac{a_{n-1}}{(n+1)(2n+3)}, \quad n = 1, 2, 3, \ldots \\
a_0 &= \text{unspecified, not equal to zero} \\
a_1 &= 0 \\
a_2 &= \frac{a_0}{2\cdot 5} \\
a_3 &= 0 \\
a_4 &= \frac{a_2}{4\cdot 9} = \frac{a_0}{2\cdot 4\cdot 5\cdot 9}
\end{aligned}
$$

Finding the pattern here is difficult, and see we only need to show there is only one solution of the form $\sum a_nx^n$, the important thing to note is that all the odd terms are zero, and the even terms all have the constant a_0 in them.

The solution we find is $y(x) = a_0\left(1 + \dfrac{x^2}{2\cdot 5} + \dfrac{x^4}{2\cdot 4\cdot 5\cdot 9} + \ldots\right)$.

So we only get one solution, and we cannot write a general solution.

This problem shows the failure of using Taylor series when looking for a series solution about a regular singular point.