Section 5.6: Regular Singular Points Part II

Example (5.6.1) Determine the exponents of the singularity for the differential equation $xy'' + 2xy' + 6e^x y = 0$ about $x_0 = 0$.

Identify $p(x) = \frac{2x}{x} = 2$ and $q(x) = \frac{6e^x}{x}$.

Since $p(x)$ is not analytic at $x_0 = 0$, we have $x_0 = 0$ as a singular point. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are both analytic at $x_0 = 0$, we have $x_0 = 0$ as a regular singular point.

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $xp(x)$ and $x^2q(x)$:

$$\begin{aligned} xp(x) &= 2x \\ &= \sum_{n=0}^{\infty} p_n x^n \\ &= p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \end{aligned}$$

so $p_0 = 0$ (the only nonzero coefficient is $p_1 = 2$).

$$\begin{aligned} x^2q(x) &= 6xe^x \\ &= 6x \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 6x + 6x^2 + 6\frac{x^3}{2} + \cdots \\ &= \sum_{n=0}^{\infty} q_n x^n \\ &= q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots \end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $xp(x) \sim p_0$ and $x^2q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned} xy'' + 2xy' + 6e^x y &= 0 \\ x^2y'' + x \cdot 2xy' + 6xe^x y &= 0 \\ x^2y'' + x \cdot p_0 y' + q_0 y &= 0 \quad \text{associated Euler equation} \\ x^2y'' + x \cdot (0)y' + (0)y &= 0 \\ x^2y'' &= 0 \end{aligned} \tag{1}$$

This can be solved by assuming $y = x^r$; $y'' = r(r-1)x^{r-2}$, so substituting into Eq. (1),

$$\begin{aligned} x^2 r(r-1)x^r &= 0 \\ r(r-1) &= 0 \quad \text{indicial equation} \end{aligned}$$

So the exponents at the singularity are $r_1 = 0$ and $r_2 = 1$.

Example (5.6.11) Find the exponents at the singularity for all the regular singular points of the differential equation $(4 - x^2)y'' + 2xy' + 3y = 0$.

First, we need to find the regular singular points.

Identify $p(x) = \dfrac{2x}{4-x^2} = \dfrac{2x}{(2-x)(2+x)}$ and $q(x) = \dfrac{3}{4-x^2} = \dfrac{3}{(2-x)(2+x)}$.

Since $p(x)$ is not analytic at $x_0 = \pm 2$, we have $x_0 = \pm 2$ as singular points. These are also the singular points for $q(x)$.

Consider $x = +2$:

Since $(x-2)p(x) = -\dfrac{2x}{2+x}$ and $(x-2)^2 q(x) = -\dfrac{3(x-2)}{2+x}$ are both analytic at $x_0 = +2$, we have $x_0 = +2$ as a regular singular point.

Consider $x = -2$:

Since $(x+2)p(x) = \dfrac{2x}{2-x}$ and $(x+2)^2 q(x) = \dfrac{3(x+2)}{2-x}$ are both analytic at $x_0 = -2$, we have $x_0 = -2$ as a regular singular point.

OK, now we need to determine the exponents at the singularity for each regular singular point.

Consider $x = +2$:

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $(x-2)p(x)$ and $(x-2)^2 q(x)$:

$$\begin{aligned}
(x-2)p(x) &= -\frac{2x}{2+x} \\
&= -1 - \frac{1}{4}(x-2) + \frac{1}{16}(x-2)^2 + \cdots \quad \text{Taylor series about } x_0 = -2 \\
&= p_0 + p_1(x-2) + p_2(x-2)^2 + p_3(x-2)^3 + \cdots
\end{aligned}$$

so $p_0 = -1$.

$$\begin{aligned}
(x-2)^2 q(x) &= -\frac{3(x-2)}{2+x} \\
&= 0 - \frac{3}{4}(x-2) + \frac{3}{16}(x-2)^3 - \cdots \quad \text{Taylor series about } x_0 = -2 \\
&= q_0 + q_1(x-2) + q_2(x-2)^2 + q_3(x-2)^3 + \cdots
\end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $(x-2)p(x) \sim p_0$ and $(x-2)^2 q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned}
(4-x^2)y'' + 2xy' + 3y &= 0 \\
y'' + \frac{2x}{(2-x)(2+x)}y' + \frac{3}{(2-x)(2+x)}y &= 0 \\
(x-2)^2 y'' + (x-2)\cdot(x-2)\frac{2x}{(2-x)(2+x)}y' + (x-2)^2\frac{3}{(2-x)(2+x)}y &= 0 \\
(x-2)^2 y'' - (x-2)\cdot\frac{2x}{2+x}y' - \frac{3(x-2)}{2+x}y &= 0 \\
(x-2)^2 y'' + (x-2)p_0 y' + q_0 y &= 0 \quad \text{associated Euler equation} \\
(x-2)^2 y'' + (x-2)(-1)y' + (0)y &= 0 \\
(x-2)^2 y'' - (x-2)y' &= 0 \qquad (2)
\end{aligned}$$

This can be solved by assuming $y = (x-2)^r$; $y' = r(x-2)^{r-1}$, $y'' = r(r-1)(x-2)^{r-2}$, so substituting into Eq. (2),

$$\begin{aligned}
(x-2)^2 r(r-1)(x-2)^{r-2} - (x-2)r(x-2)^{r-1} &= 0 \\
r(r-1) - r &= 0 \quad \text{indicial equation} \\
r(r-2) &= 0
\end{aligned}$$

So the exponents at the singularity $x_0 = -2$ are $r_1 = 0$ and $r_2 = 2$.

Consider $x = -2$:

The exponents of the singularity are the solutions to the indicial equation, and the indicial equation can be found from the associated Euler equation. We need the Taylor series expansions of $(x+2)p(x)$ and $(x+2)^2 q(x)$:

$$\begin{aligned}
(x+2)p(x) &= \frac{2x}{2-x} \\
&= -1 + \frac{1}{4}(x+2) + \frac{1}{16}(x+2)^2 + \cdots \quad \text{Taylor series about } x_0 = +2 \\
&= p_0 + p_1(x+2) + p_2(x+2)^2 + p_3(x+2)^3 + \cdots
\end{aligned}$$

so $p_0 = -1$.

$$\begin{aligned}
(x+2)^2 q(x) &= \frac{3(x+2)}{2-x} \\
&= 0 + \frac{3}{4}(x+2) + \frac{3}{16}(x+2)^3 - \cdots \quad \text{Taylor series about } x_0 = +2 \\
&= q_0 + q_1(x+2) + q_2(x+2)^2 + q_3(x+2)^3 + \cdots
\end{aligned}$$

so $q_0 = 0$.

The associated Euler equation replaces $(x+2)p(x) \sim p_0$ and $(x+2)^2 q(x) \sim q_0$, so our equation becomes:

$$\begin{aligned}
(4-x^2)y'' + 2xy' + 3y &= 0 \\
y'' + \frac{2x}{(2-x)(2+x)}y' + \frac{3}{(2-x)(2+x)}y &= 0 \\
(x+2)^2 y'' + (x+2)\cdot(x+2)\frac{2x}{(2-x)(2+x)}y' + (x+2)^2\frac{3}{(2-x)(2+x)}y &= 0 \\
(x+2)^2 y'' + (x+2)\cdot\frac{2x}{2-x}y' + \frac{3(x+2)}{2-x}y &= 0 \\
(x+2)^2 y'' + (x+2)p_0 y' + q_0 y &= 0 \quad \text{associated Euler equation} \\
(x+2)^2 y'' + (x+2)(-1)y' + (0)y &= 0 \\
(x+2)^2 y'' - (x+2)y' &= 0 \qquad (3)
\end{aligned}$$

This can be solved by assuming $y = (x+2)^r$; $y' = r(x+2)^{r-1}$, $y'' = r(r-1)(x+2)^{r-2}$, so substituting into Eq. (3),

$$\begin{aligned}
(x+2)^2 r(r-1)(x+2)^{r-2} - (x+2)r(x+2)^{r-1} &= 0 \\
r(r-1) - r &= 0 \quad \text{indicial equation} \\
r(r-2) &= 0
\end{aligned}$$

So the exponents at the singularity $x_0 = -2$ are $r_1 = 0$ and $r_2 = 2$.

If we can remember the following form, we can get the indicial equation directly from $F(r) = r(r-1) + p_0 r + q_0$, which is the from of the indicial equation for the associated Euler equation. If we forget it, we can use the process described in the solutions to create and solve the associated Euler equation.