Section 6.1 Defnition of Laplace Transforms

Example (6.1.2) See *Mathematica* file.

Example (6.1.5) Find the Laplace transform of the following functions

(a) t (b) t^2 (c) t^n

(a)

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty te^{-st}\, dt \end{aligned}$$

use parts to do the integral, $\int u\, dv = uv - \int v\, du$: $u = t, dv = e^{-st}\, dt, \quad du = dt, v = -e^{-st}/s$

$$\begin{aligned} &= -\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt \\ &= -\lim_{t\to\infty} \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_0^\infty \end{aligned}$$

the limit is zero if $s > 0$ (using l'Hospital's rule)

$$\begin{aligned} &= -0 + \frac{1}{s^2} + 0 \\ \mathcal{L}[t] &= \frac{1}{s^2}, \quad s > 0 \end{aligned}$$

(b) We will use the result from (a) in part (b).

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty t^2 e^{-st}\, dt \end{aligned}$$

use parts to do the integral, $\int u\, dv = uv - \int v\, du$: $u = t^2, dv = e^{-st}\, dt, \quad du = 2tdt, v = -e^{-st}/s$

$$\begin{aligned} &= -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\, dt \\ &= -\lim_{t\to\infty} \frac{t^2}{s}e^{-st} + \frac{2}{s}\frac{1}{s^2} \end{aligned}$$

the limit is zero if $s > 0$ (using l'Hospital's rule twice)

$$\begin{aligned} &= -0 + \frac{2}{s^3} \\ \mathcal{L}[t^2] &= \frac{2}{s^3}, \quad s > 0 \end{aligned}$$

(c) This contains both (a) and (b) for specific values of n.

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty t^n e^{-st}\, dt \end{aligned}$$

use parts to do the integral, $\int u\,dv = uv - \int v\,du$: $u = t^n, dv = e^{-st}\,dt, \quad du = nt^{n-1}dt, v = -e^{-st}/s$

$$\begin{aligned}
&= -\frac{t^n}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt \\
&= -\lim_{t\to\infty}\frac{t^n}{s}e^{-st} + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt
\end{aligned}$$

the limit is zero if $s > 0$ (using l'Hospital's rule n times)

$$\begin{aligned}
&= -0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt \\
&= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt
\end{aligned}$$

Now, use this result again.

$$\begin{aligned}
&= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}\,dt \\
&= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\int_0^\infty t^{n-3}e^{-st}\,dt \\
&\vdots \\
&= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= \frac{n!}{s^n}\int_0^\infty e^{-st}\,dt \\
&= -\frac{n!}{s^n}\cdot\frac{1}{s}e^{-st}\Big|_0^\infty \\
\mathcal{L}[t^n] &= \frac{n!}{s^{n+1}}, \quad s > 0
\end{aligned}$$

Example (6.1.6) Find the Laplace transform of $f(t) = \cos at$, where $a \in \mathbb{R}$.

$$\begin{aligned}
F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^\infty \cos at\, e^{-st}\,dt
\end{aligned}$$

use parts twice to do the integral, $\int u\,dv = uv - \int v\,du$: $u = \cos at, dv = e^{-st}\,dt, \quad du = -a\sin at\,dt, v = -e^{-st}/s$

$$= -\frac{\cos at}{s}e^{-st}\Big|_0^\infty + \frac{a}{s}\int_0^\infty \sin at\, e^{-st}\,dt$$

when evaluating the first term we require $s > 0$

$$= \frac{1}{s} + \frac{a}{s}\int_0^\infty \sin at\, e^{-st}\,dt$$

use parts again, $\int u\,dv = uv - \int v\,du$: $u = \sin at, dv = e^{-st}\,dt, \quad du = a\cos at\,dt, v = -e^{-st}/s$

$$
\begin{aligned}
&= \frac{1}{s} + \frac{a}{s}\left(-\frac{\sin at}{s}e^{-st}\Big|_0^\infty - \frac{a}{s}\int_0^\infty \cos at\, e^{-st}\,dt\right) \\
&= \frac{1}{s} - \frac{a^2}{s^2}\int_0^\infty \cos at\, e^{-st}\,dt \\
F(s) &= \frac{1}{s} - \frac{a^2}{s^2}F(s) \\
F(s) &= \frac{s}{s^2+a^2}, \quad s > 0 \\
\mathcal{L}[\cos at] &= \frac{s}{s^2+a^2}, \quad s > 0
\end{aligned}
$$