Section 6.2 Solution of Initial Value Problem

Finding the inverse Laplace transform requires the use of the table of Laplace transforms in the text.

Example (6.2.5) Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+5}$.

Since the denominator is quadratic, we should complete the square on it and then try to use one of the forms from Table 6.2.1.

$$\begin{aligned}
s^2+2s+5 &= s^2+2s+1-1+5 \\
&= (s+1)^2+4 \\
\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] &= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right]
\end{aligned}$$

Use Table 6.2.1 # 10 with $a=-1$ and $b=2$

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t}\cos 2t, \quad s>-1$$

Example (6.2.10) Find the inverse Laplace transform of $\dfrac{2s-3}{s^2+2s+10}$.

Since the denominator is quadratic, we should complete the square on it and then try to use one of the forms from Table 6.2.1.

$$\begin{aligned}
s^2+2s+10 &= s^2+2s+1-1+10 \\
&= (s+1)^2+9 \\
\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] &= \mathcal{L}^{-1}\left[\frac{2s-3}{(s+1)^2+9}\right]
\end{aligned}$$

Now, we need to work on the numerator. We want to be able to use Table 6.2.1 # 9 and # 10.

$$\begin{aligned}
2s-3 &= 2\left(s-\frac{3}{2}\right) \\
&= 2\left(s+1-\frac{5}{2}\right) \\
&= 2(s+1)-5
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] &= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] - \mathcal{L}^{-1}\left[\frac{5}{(s+1)^2+9}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] - \frac{5}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+9}\right] \\
&= 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t
\end{aligned}$$

Example (6.2.27 (c)) The Bessel fucntion of the first kind has the Taylor series $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$.

Assuming the Laplace transforms can be computed term by term, verify that

$$\mathcal{L}[J_0(t)] = (s^2+1)^{-1/2}, \quad s > 1,$$
$$\mathcal{L}[J_0(\sqrt{t})] = s^{-1}e^{-1/(4s)}, \quad s > 0.$$

$$\begin{aligned}\mathcal{L}[J_0(t)] &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \int_0^{\infty} t^{2n} e^{-st}\, dt\end{aligned}$$

This integral can be worked out using parts $2n$ times (similar to how Homework 6.1.5(c) was done), or you can use *Mathematica* to determine:

$$\int_0^{\infty} t^{2n} e^{-st}\, dt = \frac{(2n)!}{s^{2n+1}}, \quad s > 0.$$

Therefore, we have

$$\mathcal{L}[J_0(t)] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2 s^{2n+1}}$$

We can use *Mathematica* to help us recognize this Taylor series.

$$\mathcal{L}[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$$

For the second part of the question, the we get the following:

$$\begin{aligned}\mathcal{L}[J_0(\sqrt{t})] &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n}}{2^{2n}(n!)^2}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \int_0^{\infty} t^{n} e^{-st}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^{2n}(n!)^2 2^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^{2n} s^{n+1}} \\ &= \frac{1}{s} e^{-1/(4s)}\end{aligned}$$

which is again arrived at with some assistance from *Mathematica*.