Section 6.4 Discontinuous Forcing Functions

Example (6.4.3) Solve the IVP $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi), \quad y'(0) = y(0) = 0.$

Take Laplace transform:

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\sin t] - \mathcal{L}[u_{2\pi}(t)\sin(t - 2\pi)]$$

Use Table 6.2.1:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}$$
$$s^2Y(s) + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1 - e^{-2\pi s}}{(s^2+1)(s^2+4)}$$

Take the inverse Laplace Transform:

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{1 - e^{-2\pi s}}{(s^2+1)(s^2+4)}\right]$$

Use *Mathematica* to do the partial fractions: `Apart[1/(s^2+4)/(s^2+1)]`

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+1}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{e^{-2\pi s}}{s^2+4}\right]$$

Use Table 6.2.1:

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\sin(t - 2\pi) + \frac{1}{6}u_{2\pi}(t)\sin(2(t - 2\pi))$$

Simplify:

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - \frac{1}{3}u_{2\pi}(t)\sin t + \frac{1}{6}u_{2\pi}(t)\sin 2t$$
$$y(t) = \begin{cases} \frac{1}{3}\sin t - \frac{1}{6}\sin 2t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$$

See the *Mathematica* file for plots of the solution and forcing function.

Example (6.4.11) Solve the IVP $y'' + 4y = u_{\pi}(t) - u_{3\pi}(t), \quad y'(0) = y(0) = 0.$

Take Laplace transform:

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[u_{\pi}(t)] - \mathcal{L}[u_{3\pi}(t)]$$

Use Table 6.2.1:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}$$
$$s^2Y(s) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-3\pi s}}{s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}$$

Take the inverse Laplace Transform:

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2+4)}\right]$$

Use *Mathematica* to do the partial fractions: `Apart[1/s/(s^2+4)]`

$$y(t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{e^{-\pi s}s}{s^2+4}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{e^{-3\pi s}}{s}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{e^{-3\pi s}s}{s^2+4}\right]$$

Use Table 6.2.1:

$$y(t) = \frac{1}{4}u_{\pi}(t) - \frac{1}{4}u_{\pi}(t)\cos 2t - \frac{1}{4}u_{3\pi}(t) + \frac{1}{4}u_{3\pi}(t)\cos 2t$$

Simplify:

$$y(t) = \begin{cases} 0, & 0 < t < \pi \\ \frac{1}{4} - \frac{1}{4}\cos 2t, & \pi < t < 3\pi \\ 0, & t > 3\pi \end{cases}$$

See the *Mathematica* file for plots of the solution and forcing function.