Section 6.5 Impulse Functions

Example (6.5.5) Solve the IVP $y'' + 2y' + 3y = \sin t - \delta(t - 3\pi), \quad y'(0) = y(0) = 0.$

Take Laplace transform:

$$\mathcal{L}[y''] + 2\mathcal{L}[y'] + 3\mathcal{L}[y] = \mathcal{L}[\sin t] - \mathcal{L}[\delta(t - 3\pi)]$$

Use Table 6.2.1:

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y'(0) + 3Y(s) = \frac{1}{s^2+1} - e^{-3\pi s}$$
$$s^2Y(s) + 2sY(s) + 3Y(s) = \frac{1}{s^2+1} - e^{-3\pi s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} - \frac{e^{-3\pi s}}{s^2+2s+3}$$

Take the inverse Laplace Transform:

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2+1)(s^2+2s+3)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3\pi s}}{s^2+2s+3}\right]$$

Use *Mathematica* to do the partial fractions: `Apart[1/(s^2+1)/(s^2+2s+3)].`
We will also need to complete the square on $s^2 + 2s + 3 = (s+1)^2 + 2$.

$$y(t) = -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{(s+1)^2+2}\right] + \frac{1}{4\sqrt{2}}\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+1)^2+2}\right] - \mathcal{L}^{-1}\left[\frac{e^{-3\pi s}}{(s+1)^2+2}\right]$$

Hopefully you've been able to follow to here. Let's treat each application of Table 6.2.1 in turn, so we can see what's going on.

Use #6:

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$$

Use #5:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

Use #9 and #10:

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{(s+1)^2+2}\right] &= \mathcal{L}^{-1}\left[\frac{s+1-1}{(s+1)^2+2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+2} - \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2}\right] \\ &= e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t\end{aligned}$$

Use #10:

$$\mathcal{L}^{-1}\left[\frac{\sqrt{2}}{(s+1)^2+2}\right] = e^{-t}\sin\sqrt{2}t$$

Use #13 and #9:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{e^{-3\pi s}}{(s+1)^2+2}\right] &= \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left[\frac{e^{-3\pi s}\sqrt{2}}{(s+1)^2+2}\right] \\
&= \frac{1}{\sqrt{2}}u_{3\pi}(t)e^{-(t-3\pi)}\sin(\sqrt{2}(t-3\pi)) \\
&= \frac{1}{\sqrt{2}}u_{3\pi}(t)e^{-t+3\pi}\sin(\sqrt{2}t-3\sqrt{2}\pi)
\end{aligned}$$

Putting this all back, we see that the solution is

$$\begin{aligned}
y(t) &= -\frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{4}\left(e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}t\right) + \frac{1}{4\sqrt{2}}e^{-t}\sin\sqrt{2}t - \frac{1}{\sqrt{2}}u_{3\pi}(t)e^{-t+3\pi}\sin(\sqrt{2}t-3\sqrt{2}\pi) \\
&= -\frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{4}e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}u_{3\pi}(t)e^{-t+3\pi}\sin(\sqrt{2}t-3\sqrt{2}\pi) \\
&= \begin{cases} -\frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{4}e^{-t}\cos\sqrt{2}t, & 0 < t < 3\pi \\ -\frac{1}{4}\cos t + \frac{1}{4}\sin t + \frac{1}{4}e^{-t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-t+3\pi}\sin(\sqrt{2}t-3\sqrt{2}\pi), & t \geq 3\pi \end{cases}
\end{aligned}$$

See the *Mathematica* file for plots of the solution.

Example (6.5.14) Solve the IVP $y'' + \gamma y' + y = \delta(t-1), \quad y'(0) = y(0) = 0$ where γ is the damping coefficietnt.

Find the time when the solution attains its maximum value as a function of γ.

Sketch the situation using *Mathematica*'s `Manipulate` command.

Take Laplace transform:

$$\mathcal{L}[y''] + \gamma\mathcal{L}[y'] + \mathcal{L}[y] = \mathcal{L}[\delta(t-1)]$$

Use Table 6.2.1:

$$\begin{aligned}
&s^2Y(s) - sy(0) - y'(0) + \gamma sY(s) - \gamma y'(0) + Y(s) = e^{-s} \\
&s^2Y(s) + \gamma sY(s) + Y(s) = e^{-s}
\end{aligned}$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-s}}{s^2+\gamma s+1}$$

Take the inverse Laplace Transform:

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2+\gamma s+1}\right]$$

We will also need to complete the square on $s^2 + \gamma s + 1 = (s + \gamma/2)^2 + 1 - \gamma^2/4$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{e^{-s}}{(s+\gamma/2)^2 + 1 - \gamma^2/4}\right] \\ &= \frac{1}{\sqrt{1-\gamma^2/4}}\mathcal{L}^{-1}\left[\frac{e^{-s}\sqrt{1-\gamma^2/4}}{(s+\gamma/2)^2 + 1 - \gamma^2/4}\right] \end{aligned}$$

Use Table 6.2.1 #13 and #9

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{1-\gamma^2/4}} u_1(t) e^{-\gamma(t-1)/2} \sin(\sqrt{1-\gamma^2/4}(t-1)) \\ &= \begin{cases} 0, & 0 < t < 1 \\ \frac{1}{\sqrt{1-\gamma^2/4}} e^{-\gamma(t-1)/2} \sin(\sqrt{1-\gamma^2/4}(t-1)), & t > 1 \end{cases} \end{aligned}$$

We see that our solution is not valid for $\gamma = 2$, since we would get an indeteminant form. Let's determine the solution when $\gamma = 2$.

$$\lim_{\gamma\to 2} y(t) = \lim_{\gamma\to 2} \frac{1}{\sqrt{1-\gamma^2/4}} u_1(t) e^{-\gamma(t-1)/2} \sin(\sqrt{1-\gamma^2/4}(t-1))$$

Use l'Hospital's rule:

$$\begin{aligned} \lim_{\gamma\to 2} y(t) &= u_1(t) e^{-(t-1)} \lim_{\gamma\to 2} \frac{4\sqrt{1-\gamma^2/4}}{\gamma} \frac{\gamma(t-1)}{4\sqrt{1-\gamma^2/4}} \cos(\sqrt{1-\gamma^2/4}(t-1)) \\ &= u_1(t) e^{-(t-1)}(t-1) \end{aligned}$$

Probably the best way tio presenty our solution is as follows:

$$y(t) = \begin{cases} \frac{1}{\sqrt{1-\gamma^2/4}} u_1(t) e^{-\gamma(t-1)/2} \sin(\sqrt{1-\gamma^2/4}(t-1)), & \gamma \neq 2, \\ u_1(t) e^{-(t-1)}(t-1), & \gamma = 2. \end{cases}$$

See the *Mathematica* file for plots of the solution.