Section 7.1 Introduction to Systems of Differential Equations

Example (7.1.1) Transform the given equation into a system of first order equations.

$$u'' + \frac{1}{2}u' + 2u = 0$$

We want a system in the two unknown functions u_1 and u_2 (it was my choice to use u_1 and u_2, you can use whatever you want).

Let

$$\begin{aligned} u_1 &= u \\ u_2 &= u_1' \qquad (1) \end{aligned}$$

Therefore

$$\begin{aligned} u_2 &= u' \\ u_2' &= u'' \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} u'' + \frac{1}{2}u' + 2u &= 0 \\ u_2' + \frac{1}{2}u_2 + 2u_1 &= 0 \\ u_2' &= -\frac{1}{2}u_2 - 2u_1 \qquad (2) \end{aligned}$$

The system of differential equations is therefore given by Eqs. (1)–(2):

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -\frac{1}{2}u_2 - 2u_1 \end{aligned}$$

Solving this system is equivalent to solving the original second order differential equation.

Example (7.1.4) Transform the given equation into a system of first order equations.

$$u'''' - u = 0$$

We want a system in the four unknown functions u_1, u_2, u_3, and u_4.

Let

$$\begin{aligned} u_1 &= u \\ u_2 &= u_1' \qquad (3) \\ u_3 &= u_2' \qquad (4) \\ u_4 &= u_3' \qquad (5) \end{aligned}$$

Therefore

$$\begin{aligned} u_2 &= u_1' = u' \\ u_3 &= u_2' = u'' \\ u_4 &= u_3' = u''' \\ u_4' &= u'''' \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} u'''' - u = 0 &= 0 \\ u_4' - u_1 &= 0 \\ u_4' &= u_1 \end{aligned} \tag{6}$$

The system of differential equations is therefore

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= u_4 \\ u_4' &= u_1 \end{aligned}$$

Solving this system is equivalent to solving the original fourth order differential equation.

Example (7.1.6) Transform the given initial value problem into an initial value problem for two first order equations.

$$u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, u'(0) = u_0'$$

This problem will show us that any second order linear IVP can be represented as an initial value problem for a system consisting of a system of first order equations.

Let

$$\begin{aligned} u_1 &= u \\ u_2 &= u_1' \end{aligned} \tag{7}$$

Therefore

$$\begin{aligned} u_2 &= u' \\ u_2' &= u'' \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} u'' + p(t)u' + q(t)u &= g(t) \\ u_2' + p(t)u_2 + q(t)u_1 &= g(t) \\ u_2' &= -p(t)u_2 - q(t)u_1 + g(t) \end{aligned} \tag{8}$$

The system of differential equations is therefore given by Eqs. (7)–(8):

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= -p(t)u_2 - q(t)u_1 + g(t) \end{aligned}$$

The initial conditions are

$$\begin{aligned} u_1(0) &= u(0) = u_0 \\ u_2(0) &= u'(0) = u_0' \end{aligned}$$

Example (7.1.8) Transform the given IVP system into a single equation IVP of second order. Sketch the graph in the x_1x_2-plane.

$$x_1' = 3x_1 - 2x_2, \tag{9}$$
$$x_2' = 2x_1 - 2x_2, \tag{10}$$
$$x_1(0) = 3, \quad x_2(0) = 1/2$$

We need to eliminate one of the variables. Let's eliminate x_1. That means we will be trying to create a second order differential equation in x_2. Let's relabel $y = x_2$, just to help us identify the new second order differential equation a bit easier.

OK, so we know we want a second order differential equation in $y = x_2$. That means we will want to get x_2'' involved somehow. We can do that by differentiating Eq. (10).

$$\frac{d}{dt}\left[x_2' = 2x_1 - 2x_2\right]$$
$$x_2'' = 2x_1' - 2x_2'$$
$$y'' = 2x_1' - 2y'$$

Now, we need to eliminate the x_1'. We can use Eq. (9) to do this.

$$y'' = 2x_1' - 2y'$$
$$y'' = 2(3x_1 - 2x_2) - 2y'$$
$$y'' = 6x_1 - 4y - 2y'$$

Now, we need to eliminate the x_1. We can use the Eq. (10) to do this (yes, we use this equation *again!*)

$$y'' = 3(x_2' + 2x_2) - 4y - 2y'$$
$$y'' = 3(y' + 2y) - 4y - 2y'$$
$$y'' = 3y' + 6y - 4y - 2y'$$
$$y'' - y' - 2y = 0$$

The initial conditions are $y(0) = x_2(0) = 1/2$, and $y'(0) = x_2'(0) = 2x_1(0) - 2x_2(0) = 2(3) - 2(1/2) = 5$.

The corresponding initial value problem is therefore:

$$y'' - y' - 2y = 0, \quad y(0) = 1/2.y'(0) = 5.$$

Solution is assumed to be $y = e^r t$, which leads to the characteristic equation $r^2 - 4r + 8 = 0$, which means $r_1 = -1, t_2 = 2$. The general solution is therefore $y(t) = c_1e^{-t} + c_2e^{2t}$.

Apply the initial conditions to determine the constants:

$$y(0) = 1/2 \quad \longrightarrow \quad c_1 + c_2 = 1/2$$
$$y'(0) = 5 \quad \longrightarrow \quad -c_1 + 2c_2 = 5$$

This system can be solved using *Mathematica* or Cramer's rule, and you find $c_1 = -4/3$ and $c_2 = 11/6$.

The solution to the initial value problem is therefore $y(t) = -\frac{4}{3}e^{-t} + \frac{11}{6}e^{2t}$.

Relating back to x_1 and x_2, we have

$$\begin{aligned} x_2(t) &= y(t) = -\frac{4}{3}e^{-t} + \frac{11}{6}e^{2t} \\ x_1(t) &= \frac{1}{2}(x_2' + 2x_2) = \frac{1}{2}\left(\frac{4}{3}e^{-t} + \frac{11}{3}e^{2t} - \frac{8}{3}e^{-t} + \frac{11}{3}e^{2t}\right) = -\frac{2}{3}e^{-t} + \frac{11}{3}e^{2t} \end{aligned}$$

See the *Mathematica* file for sketches.