Section 7.3 Eigensystems

Example (7.3.15) Find all the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$.

First, we get the eigenvalues by solving the equation $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ \det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} &= 0 \\ (5-\lambda)(1-\lambda)+3 &= 0 \\ (\lambda-2)(\lambda-4) &= 0 \end{aligned}$$

So the eigenvalues of the matrix are $\lambda^{(1)} = 2$ and $\lambda^{(2)} = 4$.

We now get eigenvectors associated with each eigenvalue.

<u>For $\lambda^{(1)} = 2$</u>:

Solve the equation $(A - \lambda^{(1)} I)\xi = 0$.

$$\begin{aligned} (A - \lambda^{(1)} I)\xi &= 0 \\ \begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Longrightarrow 3\xi_1 - \xi_2 = 0 \text{ and } 3\xi_1 - \xi_2 = 0 \end{aligned}$$

So we have one equation in two unknowns. The systems is underdetermined.

Choose ξ_1 to be arbitrary. Let's choose $\xi_1 = 1$. Therefore, $\xi_2 = 3\xi_1 = 3(1) = 3$.

So the eigenvalue $\lambda^{(1)} = 2$ has associated eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

<u>For $\lambda^{(2)} = 4$</u>:

Solve the equation $(A - \lambda^{(2)} I)\xi = 0$.

$$\begin{aligned} (A - \lambda^{(2)} I)\xi &= 0 \\ \begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Longrightarrow \xi_1 - \xi_2 = 0 \text{ and } 3\xi_1 - 3\xi_2 = 0 \end{aligned}$$

So we have one equation in two unknowns (these equations are not independent). The systems is underdetermined.

Choose ξ_1 to be arbitrary. Let's choose $\xi_1 = 1$. Therefore, $\xi_2 = \xi_1 = 1$.

So the eigenvalue $\lambda^{(2)} = 4$ has associated eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

See the *Mathematica* file for the geometric meaning of the eigenvalues and eigenvectors for this matrix.

Example (7.3.18) Find all the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

First, we get the eigenvalues by solving the equation $\det(A - \lambda I) = 0$.

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\
\det\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix} &= 0 \\
(1-\lambda)^2 + i^2 &= 0 \\
(1-\lambda)^2 - 1 &= 0 \\
(\lambda - 0)(\lambda - 2) &= 0
\end{aligned}$$

So the eigenvalues of the matrix are $\lambda^{(1)} = 0$ and $\lambda^{(2)} = 2$.

We now get eigenvectors associated with each eigenvalue.

<u>For $\lambda^{(1)} = 0$</u>:

Solve the equation $(A - \lambda^{(1)} I)\xi = 0$.

$$\begin{aligned}
(A - \lambda^{(1)} I)\xi &= 0 \\
\begin{pmatrix} 1-0 & i \\ -i & 1-0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Longrightarrow \xi_1 + i\xi_2 = 0 \text{ and } -i\xi_1 + \xi_2 = 0
\end{aligned}$$

So we have one equation in two unknowns (these equations are not independent). The systems is underdetermined.

Choose ξ_1 to be arbitrary. Let's choose $\xi_1 = 1$. Therefore, $\xi_2 = i\xi_1 = i(1) = -i$.

So the eigenvalue $\lambda^{(1)} = 0$ has associated eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

<u>For $\lambda^{(2)} = 2$</u>:

Solve the equation $(A - \lambda^{(2)} I)\xi = 0$.

$$\begin{aligned}
(A - \lambda^{(2)} I)\xi &= 0 \\
\begin{pmatrix} 1-2 & i \\ -i & 1-2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Longrightarrow -\xi_1 + i\xi_2 = 0 \text{ and } -i\xi_1 - \xi_2 = 0
\end{aligned}$$

So we have one equation in two unknowns (these equations are not independent). The systems is underdetermined.

Choose ξ_1 to be arbitrary. Let's choose $\xi_1 = 1$. Therefore, $\xi_2 = -\xi_1 = -i$.

So the eigenvalue $\lambda^{(2)} = 2$ has associated eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Example (7.3.21) Find all the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}$.

First, we get the eigenvalues by solving the equation $\det(A - \lambda I) = 0$.

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det\left(\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) &= 0 \\
\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} &= 0 \\
\lambda^3 - 3\lambda^2 + 7\lambda - 5 &= 0
\end{aligned}$$

With some *Mathematica* assistance, we find $\lambda^{(1)} = 1$, $\lambda^{(2)} = 1 - 2i$, and $\lambda^{(3)} = 1 + 2i$. Note that the complex eigenvalues appear as complex conjugate pairs, which is not unexpected since our matrix A is real-valued.

We now get eigenvectors associated with each eigenvalue.

<u>For $\lambda^{(1)} = 1$</u>:

Solve the equation $(A - \lambda^{(1)} I)\xi = 0$.

$$\begin{aligned}
(A - \lambda^{(1)} I)\xi &= 0 \\
\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Longrightarrow 2\xi_1 - 2\xi_3 = 0 \text{ and } 3\xi_1 + 2\xi_2 = 0
\end{aligned}$$

So we have two equations in three unknowns. The systems is underdetermined. We have one arbitrary constant.

Choose ξ_3 to be arbitrary. Let's choose $\xi_3 = 1$. Therefore, $\xi_1 = \xi_3 = 1$ and $\xi_2 = -3\xi_3/2 = -3/2$.

So the eigenvalue $\lambda^{(1)} = 1$ has associated eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ -3/2 \\ 1 \end{pmatrix}$.

<u>For $\lambda^{(2)} = 1 - 2i$</u>:

Solve the equation $(A - \lambda^{(2)} I)\xi = 0$.

$$\begin{aligned}
(A - \lambda^{(1)} I)\xi &= 0 \\
\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Longrightarrow 2i\xi_1 = 0 \text{ and } 2\xi_1 + 2i\xi_2 - 2\xi_3 = 0 \text{ and } 3\xi_1 + 2\xi_2 + 2i\xi_3 = 0
\end{aligned}$$

So $\xi_1 = 0$. Therefore, we are left with

$$2i\xi_2 - 2\xi_3 = 0 \text{ and } 2\xi_2 + 2i\xi_3 = 0$$

These two equations are not independent; the first is the second times i. Therefore, we have one equation in two unknowns. The systems is underdetermined. We have one arbitrary constant.

Choose ξ_2 to be arbitrary. Let's choose $\xi_2 = 1$. Therefore, $\xi_3 = i\xi_2 = i$.

So the eigenvalue $\lambda^{(2)} = 1 - 2i$ has associated eigenvector $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$.

Since A is real-valued, we know the complex eigenvalues and eigenvectors will occur in complex conjugate pairs. Therefore, the eigenvalue $\lambda^{(3)} = 1 + 2i$ has associated eigenvector $\xi^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$.