Section 7.4 Basic Theory of Systems of 1st Order Linear Equations

Example (7.4.4) If $x_1 = y$ and $x_2 = y'$, then the second order equation

$$y' + p(t)y' + q(t)y = 0, \tag{1}$$

corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -q(t)x_1 - p(t)x_2. \end{aligned} \tag{2}$$

Show that if $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eq. (2), and if $y^{(1)}$ and $y^{(2)}$ are a fundamental set of solutions of Eq. (1), then $W(y^{(1)}, y^{(2)}) = cW(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ where c is a nonzero constant.

This problem is asking us to look at the relationship between the two ways of calculating a Wronskian we have seen in this course. Let's get on it!

Since the $y^{(1)}$ and $y^{(2)}$ form a fundamental set of solutions to Eq. (1), we know $W(y^{(1)}, y^{(2)}) \neq 0$.

$$W(y^{(1)}, y^{(2)}) = \det \begin{pmatrix} y^{(1)} & y^{(2)} \\ \left(y^{(1)}\right)' & \left(y^{(2)}\right)' \end{pmatrix} = y^{(1)} \left(y^{(2)}\right)' - y^{(2)} \left(y^{(1)}\right)'$$

Also, since $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are a fundamental set of solutions of Eq. (2), we know $W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \neq 0$.

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$$

where

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

To find the relationship between the Wronskians, we must find the relationship between the solutions.

Note that $x_1 = y$ and $x_2 = y'$, so we have $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$.

Therefore, $\mathbf{x}^{(\mathbf{1})} = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = c_1 \begin{pmatrix} y^{(1)} \\ \left(y^{(1)}\right)' \end{pmatrix}$ and $\mathbf{x}^{(\mathbf{2})} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = c_2 \begin{pmatrix} y^{(2)} \\ \left(y^{(2)}\right)' \end{pmatrix}$. Note the two solutions could differ by a nonzero constant.

So we have $y^{(1)} = c_1 x_{11}$, $y^{(2)} = c_2 x_{12}$, $\left(y^{(1)}\right)' = c_1 x_{21}$, and $\left(y^{(2)}\right)' = c_2 x_{22}$. Therefore,

$$\begin{aligned} W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) &= x_{11}x_{22} - x_{12}x_{21} \\ &= \frac{1}{c_1 c_2} \left(y^{(1)} \left(y^{(2)}\right)' - y^{(2)} \left(y^{(1)}\right)' \right) \\ &= cW(y^{(1)}, y^{(2)}) \end{aligned}$$