Bessel's Equation

There are many special functions which arise as solutions to differential equations (Hermite, Legendre, Chebyshev, etc.). Here we will look at how one important class of functions, Bessel Functions, arise through a series solution to a differential equation. Bessel Functions of the First Kind are particularly important in the study of partial differential equations, and arise in the study of vibrating circular drumheads, heat equations, and many other areas where cylindrical symmetry is present.

Bessel's equation is a differential equation of the form

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

where v is a real number.

The point $x = 0$ is a singular point, since $p(x) = \dfrac{1}{x}$ and $q(x) = 1 - \dfrac{v^2}{x^2}$ are not analytic at $x = 0$. Recall that analytic means there exists a Taylor series of the function about the point with nonzero radius of convergence.

Further, since $xp(x) = 1$ and $x^2q(x) = x^2 - v^2$ are analytic at $x = 0$, the point $x = 0$ is a regular singular point.

Therefore, we can try to determine at least one series solution of Bessel's equation which has the form $y = \sum_{n=0}^{\infty} a_n x^{r+n}$.

Bessel Functions of the First Kind

Differentiating $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ and substituting into Bessel's equation, we find

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \qquad y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}, \qquad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2},$$

$$\begin{aligned}
x^2\left(\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}\right) + x\left(\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}\right) + (x^2 - v^2)\left(\sum_{n=0}^{\infty} a_n x^{r+n}\right) &= 0 \\
\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} v^2 a_n x^{r+n} &= 0 \\
\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} - \sum_{n=0}^{\infty} v^2 a_n x^{r+n} &= 0 \\
\left(\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n} + \sum_{n=0}^{\infty}(r+n)a_n x^{n} + \sum_{n=2}^{\infty} a_{n-2} x^{n} - \sum_{n=0}^{\infty} v^2 a_n x^{n}\right) x^r &= 0 \\
\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n} + \sum_{n=0}^{\infty}(r+n)a_n x^{n} + \sum_{n=2}^{\infty} a_{n-2} x^{n} - \sum_{n=0}^{\infty} v^2 a_n x^{n} &= 0 \\
\left[r(r-1)a_0 + ra_0 - v^2 a_0\right] x^0 + \left[(r+1)ra_1 + (r+1)a_1 - v^2 a_1\right] x^1 &+ \\
\sum_{n=2}^{\infty}(r+n)(r+n-1)a_n x^{n} + \sum_{n=2}^{\infty}(r+n)a_n x^{n} + \sum_{n=2}^{\infty} a_{n-2} x^{n} - \sum_{n=2}^{\infty} v^2 a_n x^{n} &= 0
\end{aligned}$$

$$\left[r(r-1)a_0 + ra_0 - v^2 a_0\right] x^0 + \left[(r+1)ra_1 + (r+1)a_1 - v^2 a_1\right] x^1 + \sum_{n=2}^{\infty} \left[(r+n)^2 a_n + a_{n-2} - v^2 a_n\right] x^n = 0$$

We want this to be true for all values of x. Therefore, the coefficients of powers of x must be zero. This leads to the equations:

$$\begin{aligned} r(r-1)a_0 + ra_0 - v^2 a_0 &= 0 && (1)\\ (r+1)ra_1 + (r+1)a_1 - v^2 a_1 &= 0 && (2)\\ (r+n)^2 a_n + a_{n-2} - v^2 a_n &= 0, \quad n = 2, 3, 4, \ldots && (3) \end{aligned}$$

These equations are the recursion and indicial equation. We can choose any one of them as the indicial equation; let's choose the indicial equation to be Eq. (1),

$$r(r-1)a_0 + ra_0 - v^2 a_0 = 0.$$

If we assume $a_0 \neq 0$, the solutions to the indicial equation are $r = \pm v$. We can get one solution to Bessel's equation if we choose to work with the larger root from the indicial equation, so let's assume $r = v > 0$ and proceed.

The recursions equations (Eq. (2) and (3)) become

$$\begin{aligned} (2v+1)a_1 &= 0 && (4)\\ a_n &= -\frac{a_{n-2}}{n(n+2v)}, \quad n = 2, 3, 4, \ldots && (5) \end{aligned}$$

Equation (4) tells us that $a_1 = 0$, unless $v = -1/2$, in which case the equation is satisfied regardless of the value of a_1. Since we have assumed that $v > 0$, this situation does not arise, and we may safely assume $a_1 = 0$.

We can now use Eq (5) to generate the first few a_n, and then try to determine the pattern.

$$\begin{aligned} a_0 &= \text{ unspecified, does not equal zero}\\ a_1 &= 0\\ a_2 &= -\frac{a_0}{2(2+2v)}\\ a_3 &= 0\\ a_4 &= -\frac{a_2}{4(4+2v)} = \frac{a_0}{2\cdot 4(2+2v)(4+2v)} = \frac{a_0}{2^{2\cdot 2}2!(1+v)(2+v)}\\ a_5 &= 0\\ a_6 &= -\frac{a_4}{6(6+2v)} = -\frac{a_0}{2\cdot 4\cdot 6(2+2v)(4+2v)(6+2v)} = -\frac{a_0}{2^{2\cdot 3}3!(1+v)(2+v)(3+v)} \end{aligned}$$

This is enough for us to recognize the pattern. First, let's work with $(1+v)(2+v)\cdots(m+v)$. If v is an integer, we observe that

$$\frac{1}{(1+v)(2+v)\cdots(m+v)} = \frac{1\cdot 2\cdot 3\cdots(v-1)(v)}{1\cdot 2\cdot 3\cdots(v-1)(v)(1+v)(2+v)\cdots(m+v)} = \frac{v!}{(v+m)!}.$$

If v is not an integer, we can replace the factorial with a gamma function using the relation $w! = \Gamma(1+w)$. Therefore,

$$\frac{1}{(1+v)(2+v)\cdots(m+v)} = \frac{\Gamma(1+v)}{\Gamma(1+v+m)}.$$

If you check this on *Mathematica*, you will find that this product can also be represented in terms of the Pochhammer symbol.

The general term in our expansion is therefore

$$a_{2m} = \frac{(-1)^m a_0 \Gamma(1+v)}{2^{2m} m!\, \Gamma(1+v+m)}, \quad m = 0, 1, 2, \ldots.$$

A solution to Bessel's equation is therefore

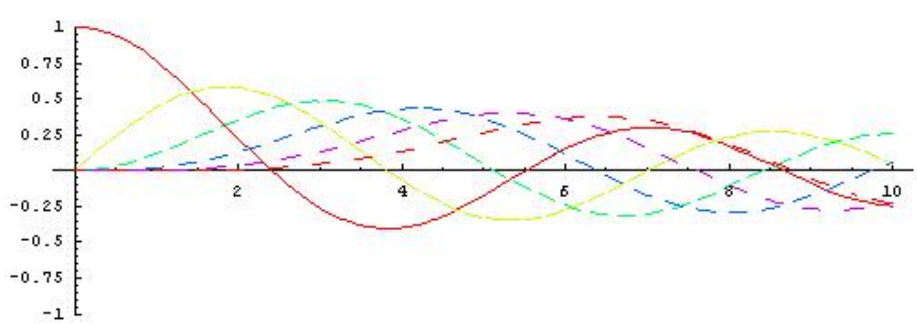
$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \sum_{m=0}^{\infty} a_{2m} x^{v+2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m a_0 \Gamma(1+v)}{2^{2m} m!\, \Gamma(1+v+m)} x^{v+2m} \\ &= a_0 x^v \Gamma(1+v) \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!\, \Gamma(1+v+m)} \end{aligned}$$

Bessel Functions of The First Kind of order v are obtained by choosing $a_0 = 1/(2^v \Gamma(1+v))$, and they are commonly denoted $J_v(x)$,

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!\, \Gamma(1+v+m)}$$

Although we assumed in our analysis that $v > 0$, we can see that$J_v(x)$ is defined for $v \in \mathbb{R}$. In fact, it can be extended to $v \in \mathbb{C}$. What is important in the solution to partial differential equations is that $J_v(x)$ is finite at $x = 0$ for integer values of v.

Here is a plot of $J_v(x)$, for $v = 0, 1, 2, 3, 4, 5$. Note that $J_0(0) = 1$.



Bessel Functions of the Second Kind

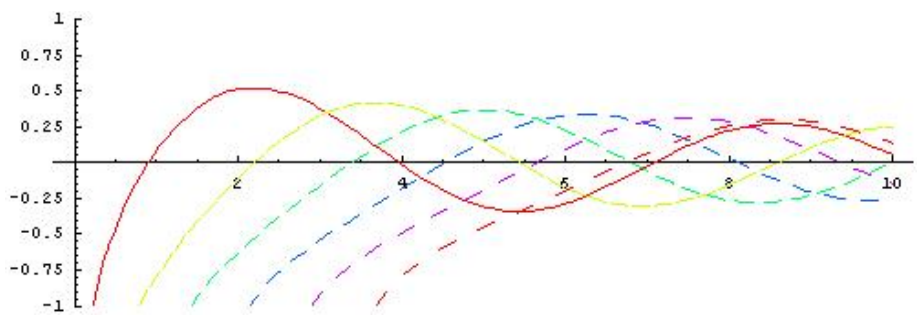
A second solution to Bessel's equation can be found using reduction of order, since we now know a first solution. Here we use the formula we derived for reduction of order earlier, where we identify $p(x) = 1/x$ from Bessel's equation.

$$\begin{aligned} y2(x) &= y_1(x) \int \frac{\exp\left(-\int p(x)\,dx\right)}{y_1^2(x)}\,dx \\ &= J_v(x) \int \frac{\exp\left(-\int (1/x)\,dx\right)}{J_v^2(x)}\,dx \\ &= J_v(x) \int \frac{1}{xJ_v^2(x)}\,dx \\ &= \frac{\pi}{2} Y_v(x) \end{aligned}$$

where the last integral is found from a book of integrals [1] or a more in-depth study of Bessel Functions than we have time for.

Notice that this procedure found a solution which is a constant multiple of $Y_v(x)$, so we can drop the constant and take as the second solution $y_2(x) = Y_v(x)$.

Here is a plot of $Y_v(x)$, for $v = 0, 1, 2, 3, 4, 5$.



Notice that $\lim_{x\to 0} Y_v(x) = -\infty$ for integer values of v. This is a logarithmic singularity that we sometimes see with second solutions to differential equations about a regular singular point. This excludes, for physical reasons, $Y_v(x)$ from solutions in many applications.

The derivation of the second solution using other techniques can be a difficult task, but realize that although our solution looks simple, we had an integral that we worked out using a table of integrals, so to understand this integral we would likely have to put in an equal amount of work.

References

[1] Gradshteyn, I. S.; Ryzhik, I. M. *Table of Integrals, Series, and Products, 6th ed.*, Academic Press, New York, **2000**. Page 664 #6.539.1