2.4 Differences Between Linear and Nonlinear Equations

Existence and Uniqueness of Solutions

Up until now, we have not thought about this at all. We have solved initial value problems and assumed that the solution was valid, and was the only solution. But we must ask the question: **Does every initial value problem have exactly one solution?**

This is a tremendously important question, if we are to use differential equations to model physical systems. We must know that the differential equation has a solution *before we spend the time and energy trying to find it.* We also want the solution to be unique if we want the solution to be a useful predictive tool.

We can answer these questions with Existence and Uniqueness theorems. We have one for linear differential equations, and one for nonlinear.

Theorem 1 (2.4.1 Linear)

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition

$$y(t_0) = y_0$$

where y_0 is an arbitrary initial value.

This is an amazing theorem–it says not only that a solution **exists**, but that solution is **unique**! It also tells you where that solution exists, in the interval over which p and g are continuous containing the initial point.

The seeds of the proof of this theorem are contained in the derivation of the solution via the integrating factor. The core idea is that we can always write the solution as

$$y = \frac{\displaystyle\int \mu(s)g(s)ds|_{s=t} + C}{\mu(t)}.$$

Although we may not be able to perform the integral in this equation, we can always write the solution formally in the above manner.

Nonlinear equations satisfy the following theorem:

Theorem 2 (2.4.2 Nonlinear)

Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \phi(t)$ of the initial value problem

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0.$$

Here is the first example we have seen that nonlinear equations are more complicated than linear ones!

Let's learn more about these theorems through specific examples. Proofs of these theorems can be found in advanced textbooks.

Example of What Theorem 2.4.1 Tells Us Find an interval in which the initial value problem

$$t\frac{dy}{dt} + 2y = 4t^2, \;\; y(1) = 2$$

has a unique solution.

Solution: Rewrite the equation in standard form:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t$$

and identify $p(t) = 2/t, g(t) = 4t$. Obviously, $g(t)$ is continuous for $t \in \mathbb{R}$; $p(t)$, however, is continuous for $t \in (-\infty, 0)$ or $t \in (0, \infty)$. The initial condition is at $t = 1 > 0$, so Theorem 2.4.1 guarantees that we will have a unique solution to the initial value problem for $t \in (0, \infty)$. We know this without solving the initial value problem!

The DE can now be solved using the integrating factor method:

$$\mu\frac{dy}{dt} + \mu\frac{2}{t}y = 4t\mu \text{ (multiply by integrating factor)} \tag{1}$$
$$\frac{d}{dt}[\mu y] = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y \text{ (product rule for derivatives)} \tag{2}$$
$$\frac{d\mu}{dt} = \frac{2}{t}\mu \text{ (DE for integrating factor)} \tag{3}$$
$$\int\frac{d\mu}{\mu} = 2\int\frac{dt}{t} \tag{4}$$
$$\ln|\mu| = 2\ln|t| \tag{5}$$
$$\mu = t^2 \tag{6}$$
$$t^2\frac{dy}{dt} + 2ty = 4t^3 \text{ (insert } \mu) \tag{7}$$
$$\frac{d}{dt}[t^2 y] = 4t^3 \text{ (use the product rule for derivatives to simplify)} \tag{8}$$
$$\int d[t^2 y] = \int 4t^3 dt \text{ (integrate)} \tag{9}$$
$$t^2 y = t^4 + C \tag{10}$$
$$y = t^2 + Ct^{-2}. \tag{11}$$

Applying the initial condition will pick out a single curve from the family of curves, and determine the interval for which the solution is valid. Applying $y(1) = 2$:

$$2 = 1 + C \quad \longrightarrow \quad C = 1$$

and the final, unique solution is given by: $y = t^2 + t^{-2}, \;\; t > 0$.

Notice how changing the initial condition to $y(-1) = 2$ would result in the same form for the solution but for $t < 0$.

The practical upshot of all this is that the solutions do not cross the line $t = 0$, and this discontinuity in the solutions was evident from the DE itself. We did not have to solve the IVP to uncover this discontinuity.

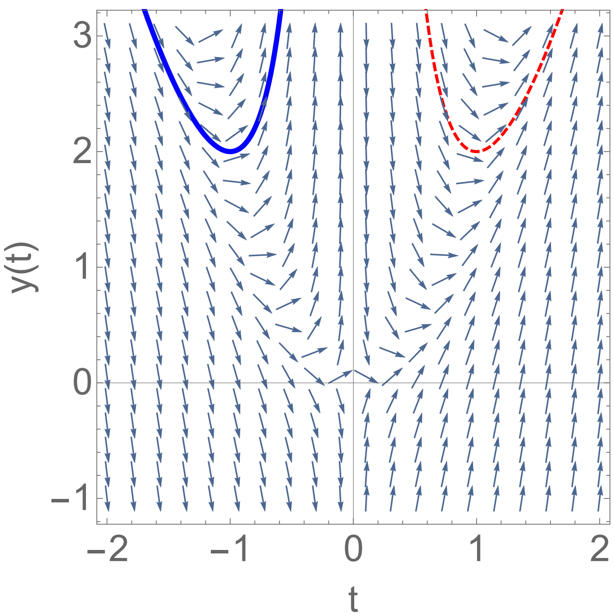

Figure 1: Solution to differential equation $t\frac{dy}{dt} + 2y = 4t^2$ with initial condition $y(1) = 2$ (red) and $y(-1) = 2$ (blue). The region the solution exists in was known before solving the initial value problem, and the solution is an explicit function.

Example of What Theorem 2.4.2 Tells Us Investigate the IVP

$$\frac{dy}{dt} = \frac{1+t^2}{3y-y^2}, \;\; y(0) = 1.$$

Note that this is nonlinear in both t and y. We must use Theorem 2.4.2.

Begin by identifying the quantities needed for Theorem 2.4.2:

$$f(t,y) = \frac{1+t^2}{3y-y^2}, \;\; \frac{\partial f}{\partial y} = -\frac{(1+t^2)(3-2y)}{(3y-y^2)^2}$$

Both these functions are continuous everywhere except when $y = 0, 3$. Consequently, we **can** draw a rectangle about the initial point $(0,1)$ where both functions are continuous, and a unique solution to the IVP will exist in this rectangle.

The size of this rectangle is not arbitrary, and we shall see that there is a further constraint which is not yet apparent. The important thing is we know a unique solution will exist in some rectangle about the initial point <u>before we solve the IVP</u>.

The DE is separable, so we shall proceed by writing it in the separable form:

$$(3y - y^2)dy = (1+t^2)dt, \tag{12}$$

$$\int (3y - y^2)dy = \int (1+t^2)dt, \tag{13}$$

$$\frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{1}{3}t^3 + C. \tag{14}$$

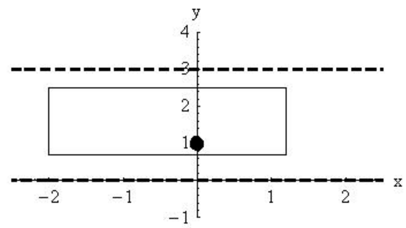

Figure 2: It is possible to draw a rectangle around the initial point where both functions are continuous (dashed line is where discontinuity occurs). Although we know from Theorem 2.4.2 a solution will exist in some region about the initial point, we do not know the size of the region. We can tell it is constrained on top at $y = 3$ and bottom at $y = 0$, but at the moment it looks like it could extend for $t \in (-\infty, \infty)$.

Let's apply the initial condition, $y(0) = 1$, and determine C.

$$\frac{3}{2} - \frac{1}{3} = C \longrightarrow C = \frac{7}{6}.$$

So the solution to the IVP is

$$\frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{1}{3}t^3 + \frac{7}{6}.$$

Since this is an implicit solution, it contains many explicit solutions, not all of which are going to be real valued.

We can use *Mathematica* to investigate further, but in general an explicit solution may not exist. Using *Mathematica*, we are able to determine the three explicit solutions, and determine which one is the solution to the IVP. The constraint $-0.912991 \leq t \leq 1.69889$ was not evident from the original differential equation. It was only discovered once we had solved the IVP and could work with determining the explicit solution to the initial value problem.

A Different Initial Condition–loss of uniqueness of solution

If the initial condition had been $y(0) = 3$, we would not have been able to draw a rectangle around the point $(0, 3)$ for which f and $\partial f/\partial y$ would be continuous; we could not use Theorem 2.4.2 to say anything about the uniqueness or existence of the solution.

But this does **not** mean a solution does not exist! It just means we can't say anything about it to begin with. In fact, with the new initial condition, we have the solution to the IVP (verify):

$$\frac{3}{2}y^2 - \frac{1}{3}y^3 = t + \frac{1}{3}t^3 + \frac{9}{2}.$$

and if you solve for the explicit solutions $y(t)$, you will find two solutions pass through the point $(0, 3)$. The solution exists, but is not unique. This is entirely due to the nonlinearity in the problem.

General Solutions. For a first order linear equation we can obtain a solution containing one arbitrary constant, and all possible solutions follow from assigning values to this constant. Thus, determining the general solution determines all possible solutions. For nonlinear equations, this is not the case! You may be able to find a solution containing an arbitrary constant, but other solutions may exist that cannot be determined from assigning a value to the constant.

Note: A General Solution is a family of curves, but a family of curves is not necessarily a General Solution!

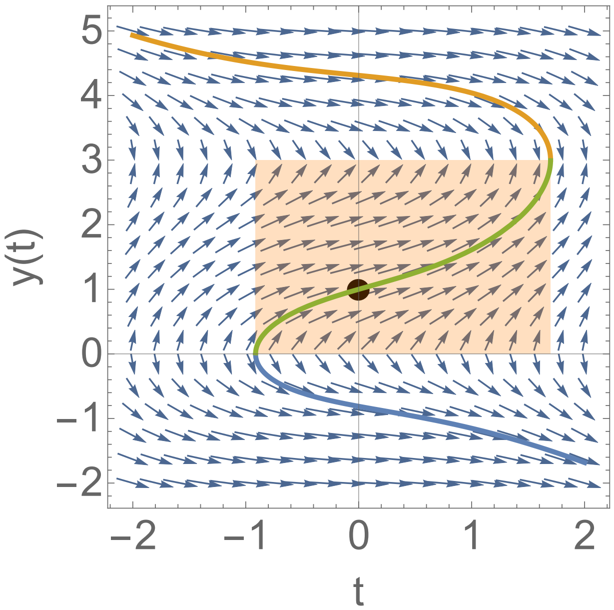

Figure 3: Solution to the initial value problem $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$, $y(0) = 1$. The rectangle indicates the region where the solution exists, and we could not determine the size of the rectangle until we had solved the initial value problem. Although in this case we did determine an explicit solution (see *Mathematica*file), in general an explicit solution may not exist. Notice the behaviour of the direction field near the three explicit solutions.

Summary of properties of linear equations.

- Assuming the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all the solutions of the DE.
- There is an explicit expression for the solution (although we may not be able to evaluate it!).
- The possible points of discontinuity of the solution can be identified without solving the problem.

None of these nice features are true for nonlinear differential equations.

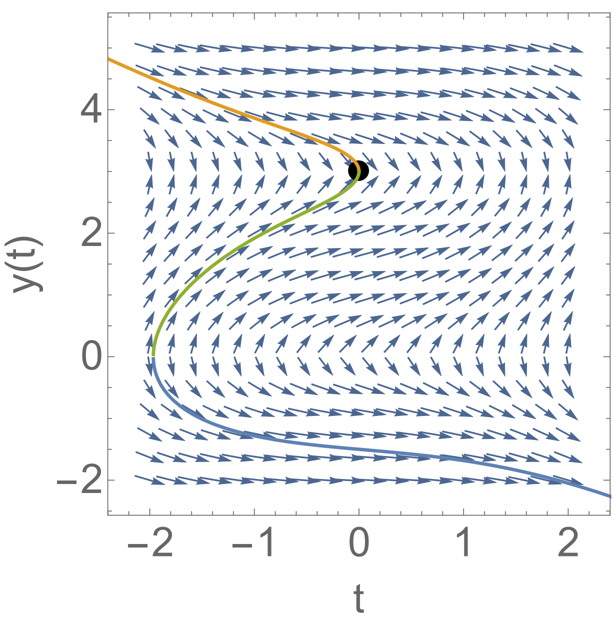

Figure 4: Solution to the initial value problem $\frac{dy}{dt} = \frac{1+t^2}{3y-y^2}$, $y(0) = 3$. In this case Theorem 2.4.2 does not apply, and although we have a solution to the IVP it is not unique.