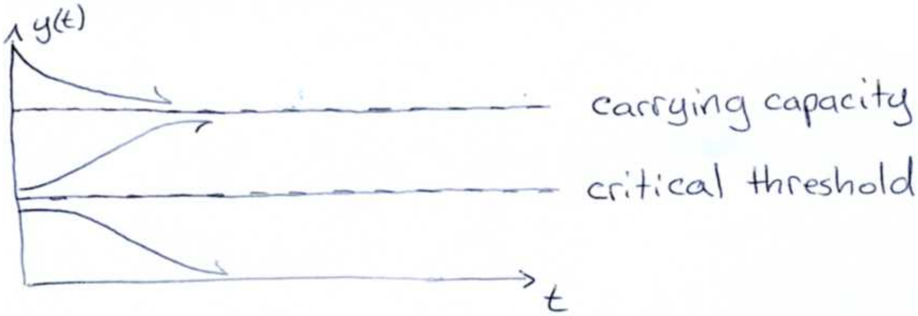
A population should

- reach a finite equilibrium (the carrying capacity) based on the resources available to the system,
- become extinct if there is an initial population below some critical amount (the critical threshold).

We also would like to have exponential growth in some limit.

So if we are modeling a population, we would like our global solution to look something like:



Armed with this knowledge, we can modify the exponential growth differential equation $\frac{dy}{dt} = ry$ to give us these features.

Consider the differential equation:

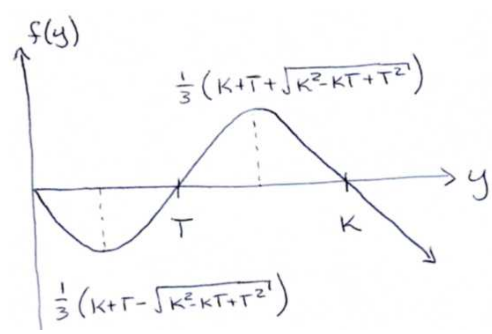
$$\frac{dy}{dt} = -ry(1-\frac{y}{T})(1-\frac{y}{K}) = f(y)$$

where $r > 0$ and $0 < T < K$. We want to verify the global solutions to this differential equation look like what we've drawn above, but without actually solving the DE.

We can easily graph $f(y)$:

- $f(y)$ is cubic, so it has at most two "humps".
- $\lim_{y\to\infty} f(y) = (-\infty)(-\infty)(-\infty) = -\infty$.
- zero at $y = 0, T, K$.
- If we solve $\frac{df}{dy} = 0$ we see the "humps" are at $y = \frac{1}{3}\left(K + T \pm \sqrt{K^2 - KT + T^2}\right)$.

  Let $y_{\min} = \frac{1}{3}\left(K + T - \sqrt{K^2 - KT + T^2}\right)$ and $y_{\max} = \frac{1}{3}\left(K + T + \sqrt{K^2 - KT + T^2}\right)$



We want a graph of $y(t)$, and we can use results from calculus and the sketch of $f(y)$ to get it!

Calculus Results: Stewart Section 4.3

The following derivative results describe the behaviour of the function $y(t)$.

- Increasing/Decreasing:
  - $y'(t) > 0$ on an interval means $y(t)$ is increasing on that interval.
  - $y'(t) < 0$ on an interval means $y(t)$ is decreasing on that interval.
  - $y'(c) = 0$ means $y(t)$ has a horizontal tangent at $t = c$.

Concave up: curve lies above its tangent or the curve is bending up.

Concave down: curve lies below its tangent or the curve is bending down.

Point of Inflection: point where curve changes concavity.

- Concave Up/Concave down:
  - $y''(t) > 0$ on an interval means $y(t)$ is concave up on that interval.
  - $y''(t) < 0$ on an interval means $y(t)$ is concave down on that interval.
  - $y''(c) = 0$ means $y(t)$ has a point of inflection at $t = c$.

Applying the Calculus to Autonomous Equations

We don't know the function $y(t)$, but we do know $y'(t) = \frac{dy}{dt} = f(y)$ from the differential equation!

Therefore, we know $f(y) = 0 \Rightarrow \frac{dy}{dt} = 0 \Rightarrow y(t)$ is an equilibrium solution.

What this means is that if we start at an initial point $y(0) = y_0$ where $f(y) = 0$, the solution will not change as time increases, so we get the constant valued solution $y(t) = y_0$.

We also know

$$f(y) > 0 \Rightarrow \frac{dy}{dt} > 0 \Rightarrow y(t) \text{ increasing} .$$

$$f(y) < 0 \Rightarrow \frac{dy}{dt} < 0 \Rightarrow y(t) \text{ decreasing} .$$

For concavity, we need to do a bit more work: $\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = \frac{d}{dy}f(y) \cdot \frac{dy}{dt} = f'(y)f(y)$.

Thus if $f'(y)$ and $f(y)$ have

- the same sign, $y(t)$ is concave up,
- the opposite sign, $y(t)$ is concave down.

From all this we can construct the solution curves $y(t)$ based on the sketch of $f(y)$.

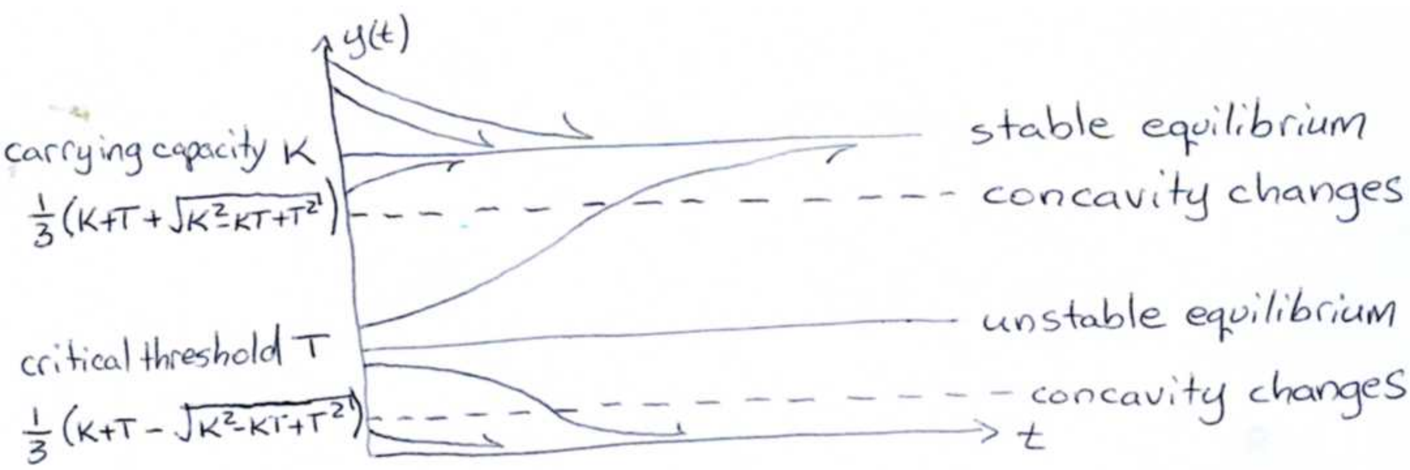
Applying the Calculus to Our Particular Problem (refer to the sketch of $f(y)$ vs y)

Since $f(y) = 0$ for $y = 0, T, K$, the equilibrium solutions are $y(t) = 0$, $y(t) = T$, $y(t) = K$.

Recall $y_{\min} = \frac{1}{3}\left(K + T - \sqrt{K^2 - KT + T^2}\right)$ and $y_{\max} = \frac{1}{3}\left(K + T + \sqrt{K^2 - KT + T^2}\right)$.

Interval	Facts From Sketch of $f(y)$	What the facts tell us about $y(t)$
$0 < y < y_{\min}$	$f'(y) < 0$ and $f(y) < 0$ (same sign)	$y(t)$ is concave up, decreasing
$y_{\min} < y < T$	$f'(y) > 0$ and $f(y) < 0$ (opposite sign)	$y(t)$ is concave down, decreasing
$T < y < y_{\max}$	$f'(y) > 0$ and $f(y) > 0$ (same sign)	$y(t)$ is concave up, increasing
$y_{\max} < y < K$	$f'(y) < 0$ and $f(y) > 0$ (opposite sign)	$y(t)$ is concave down, increasing
$y > K$	$f'(y) < 0$ and $f(y) < 0$ (same sign)	$y(t)$ is concave up, decreasing

Put it all together in a sketch:



So we see that T is the critical threshold, K is the carrying capacity. The quantity r is called the intrinsic growth rate.

Notice that the solutions approach the equilibrium solution $y(t) = K$ as $t \to \infty$ from both sides, so this is called a stable equilibrium solution.

The solutions move away from the equilibrium solution $y(t) = T$ as $t \to \infty$, so this is an unstable equilibrium solution.

You can also have semistable equilibrium solutions, which are approached from one side and moved away from on the other.

From the standpoint of someone who is using this to model a system, the global structure is very important and probably what they are most interested in. The solution to the initial value problem

$$\frac{dy}{dt} = -ry(1 - \frac{y}{T})(1 - \frac{y}{K}), \quad y(0) = y_0$$

can be found in implicit form by integration using partial fractions or *Mathematica*.

With this understanding of how the function $f(y)$ affects the global properties of the solution, one can construct other differential equations with similar properties for different $f(y)$. The text discusses many examples in the problems at the end of the section.

This is an example of an **Autonomous Differential Equation**, which is an equation where the independent variable does not appear explicitly:

$$\frac{dy}{dt} = f(y).$$

These types of equations show up naturally in population modeling problems and fluid dynamics.

The important thing with autonomous equations is that the global properties of the solution can be understood without solving the differential equation, making them extremely useful when you are searching for a suitable mathematical model. You typically know the global properties of the physical system you are studying, so you can quickly and easily construct a valid model without solving a bunch of differential equations.

Once you have the appropriate model, you can use your resources to solve the differential equation by separating, and then performing the integrals (possibly numerically):

$$\begin{aligned} \frac{dy}{f(y)} &= dt \\ \int \frac{dy}{f(y)} &= \int dt \end{aligned}$$