3.5 Nonhomogeneous Equations

The 2nd order linear **nonhomogeneous equation** with variable coefficients is

$$L[y] = y'' + p(t)y' + q(t)y = g(t),$$

where p, q, g are continuous on I.

The **associated homogeneous equation** is

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

The following theorems provide the basis for solving the nonhomogeneous general solution.

Theorem 1 3.5.1 *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation*

$$L[y](t) = y'' + p(t)y' + q(t)y = g(t),$$

then their difference is a solution of the corresponding homogeneous equation

$$L[y](t) = 0.$$

If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous equation, then

$$Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t)$$

where c_1 and c_2 are constants.

Theorem 2 3.5.2 *The general solution of the nonhomogeneous equation can be written in the form*

$$y = \phi(t) = c_1y_1(t) + c_2y_2(t) + Y(t),$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation, c_1 and c_2 are arbitrary constants, and $Y(t)$ is some **specific solution** *of the nonhomogeneous equation.*

The Process to solve second order nonhomogeneous equations:

1. Find the general solution of the corresponding homogeneous equation. This solution is called the **complementary solution**, and is denoted $y_c(t) = c_1y_1(t) + c_2y_2(t)$.

2. Find some singular solution $Y(t)$ of the nonhomogeneous equation. This is called the **particular solution**, and is denoted $y_p(t) = Y(t)$.

3. The **general solution** to the nonhomogeneous equation is given by $y(t) = y_c(t) + y_p(t)$.

Note: If you are solving an initial value problem, it is $y(t) = y_c(t) + y_p(t)$ on which the initial conditions are applied. Therefore, you must get the general solution to the nonhomogeneous equation before you use the initial conditions to determine the constants c_1 and c_2.

For homogeneous constant coefficient differential equations, we know how to determine the general solution. So we can find $y_c(t)$ for equations of the form $ay'' + by' + cy = g(t)$. To get the particular solution $y_p(t)$, we will use one of two methods (there are others): Undetermined Coefficients, or Variation of Parameters.

Example Find the general solution of the DE $u'' + w_0^2 u = \cos(wt)$, where $w_0^2 \neq w^2$.

First, solve the associated homogeneous equation $u'' + w_0^2 u = 0$.
Assume $u = e^{rt}$, since this is a constant coefficient linear differential equation.
Substitute into the DE: $r^2 e^{rt} + w_0^2 e^{rt} = 0$.
Characteristic equation: $r^2 + w_0^2 = 0$.
Roots of the characteristic equation are complex: $r = \pm w_0 = \lambda \pm \mu$. Therefore, $\lambda = 0, \mu = w_0$.
A fundamental set of solutions is $u_1 = \cos w_0 t$, $u_2 = \sin w_0 t$.
The complementary solution is therefore $u_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Get a particular solution of the nonhomogeneous equation. Assume $U(t) = A\cos wt + B \sin wt$. There is no overlap with the complementary solution, so this will work. Substitute into the DE:

$$\begin{aligned} u'' + w_0^2 u &= \cos(wt) \\ (-Aw^2 \cos wt - Bw^2 \sin wt) + w_0^2(A\cos wt + B\sin wt) &= \cos wt \\ \left[A(w_0^2 - w^2) - 1\right]\cos wt + \left[B(w_0^2 - w^2)\right]\sin wt &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of the functions of t must be zero. This gives us the two equations for the two unknowns A, B.

$$\begin{aligned} A(w_0^2 - w^2) - 1 = 0 \quad &\longrightarrow \quad A = \frac{1}{w_0^2 - w^2} \\ B(w_0^2 - w^2) = 0 \quad &\longrightarrow \quad B = 0 \end{aligned}$$

The particular solution of the nonhomogeneous DE is $y_p(t) = \dfrac{1}{w_0^2 - w^2}\cos wt$.

The general solution is $y(t) = y_c(t) + y_p(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \dfrac{1}{w_0^2 - w^2}\cos wt$.

Example Find a particular solution of the DE $u'' + w_0^2 u = \cos(w_0 t)e^t$.

First, solve the associated homogeneous equation $u'' + w_0^2 u = 0$.
This is the same as in the previous problem. The complementary solution is therefore $u_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Get a particular solution of the nonhomogeneous equation. Assume $U(t) = (\bar{A}\cos w_0 t + \bar{B}\sin w_0 t)(\bar{C}e^t) = Ae^t \cos w_0 t + Be^t \sin w_0 t$. This does not have overlap with the complementary solution, so it should be a solution of the nonhomogeneous equation.

Substitute into the DE; use *Mathematica* to help with derivatives and simplification (*Mathematica* notebook online). The differential equation becomes: $e^t \sin w_0 t(B - 2Aw_0) + e^t \cos w_0 t(A + 2Bw_0) = e^t \cos w_0 t$.

To make this true for all values of t, we equate coefficients of similar functions of t, and we have:

$$\begin{aligned} B - 2Aw_0 = 0 \quad &\Longrightarrow \text{(use Cramer's Rule; details left out)} \quad A = \frac{1}{1 + 4w_0^2} \\ A + 2Bw_0 = 1 \quad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad B = \frac{2w_0}{1 + 4w_0^2} \end{aligned}$$

The particular solution of the nonhomogeneous DE is $y_p(t) = \left(\dfrac{1}{1 + 4w_0^2}\cos w_0 t + \dfrac{2w_0}{1 + 4w_0^2}\sin w_0 t\right)e^t$.