2401 Differential Equations Section 3.6 Nonhomogeneous Equations Undetermined Coefficients

We can now solve any homogeneous second order linear differential equation with constant coefficients.

Now we look at nonhomogeneous equations.

The nonhomogeneous equation with variable coefficients is

$$L[y] = y'' + p(t)y' + q(t)y = g(t),$$

where p, q, g are continuous on I.

The associated homogeneous equation corresponding to the nonhomoegeous equation is

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

The following theorems provide the basis for solving the nonhomogeneous general solution.

Theorem. 3.6.1 *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation*

$$L[y](t) = y'' + p(t)y' + q(t)y = g(t),$$

then their difference is a solution of the corresponding homogeneous equation

$$L[y](t) = 0.$$

If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous equation, then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are constants.

The proof can be obtained by subtracting the two equations $L[Y_1](t)$ and $L[Y_2](t)$. The second statement is possible since y_1 and y_2 are a fundamental set of solutions, and any function which satisfies the DE can be expressed as a linear combination of a fundamental set of solutions. □

Theorem. 3.6.2 *The general solution of the nonhomogeneous equation can be written in the form*

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t),$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation, c_1 and c_2 are arbitrary constants, and Y is some **specific solution** *of the nonhomogeneous equation.*

The Process to solve second order nonhomogeneous equations:

1. Find the general solution of the corresponding homogeneous equation. This solution is called the *complementary* solution, and is denoted $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$.
2. Find some singular solution $Y(t)$ of the nonhomogeneous equation. This is called the *particular* solution, and is denoted $y_p(t) = Y(t)$.
3. The general solution to the nonhomogeneous equation is given by $y(t) = y_c(t) + y_p(t)$.

For homogeneous constant coefficient differential equations, we know how to determine the general solution. So we can find $y_c(t)$ for equations of the form

$$ay'' + by' + cy = g(t).$$

To get the particular solution $y_p(t)$, we will use one of two methods (there are others): Undetermined Coefficients, or Variation of Parameters.

Undetermined Coefficients

We assume that the particular solution we seek has a certain form, but with coefficients left unspecified. We substitute this assumed solution into the DE and attempt to determine the coefficients so as to satisfy the equation.

If we cannot determine the coefficients, this means there is no solution of the form we assumed. We can therefore assume another form of the solution and try again.

Pros: Easy to implement. *Cons*: Equation must be simple enough that you can virtually guess the solution before hand.

This is therefore used for cases where the associated homogeneous equation has constant coefficients and the nonhomogeneous term is from a particular set of functions (polynomials, exponentials, sines, cosines).

This may seem a drastic limitation, but many physical systems are modeled by such differential equations.

Example Find the general solution of the DE $y'' + 2y' + y = 2e^{-t}$.

First, we solve the associated homogeneous equation:

$$y'' + 2y' + y = 0.$$

Assume $y = e^{rt}$
Substitute into the DE: $r^2e^{rt} + 2re^{rt} + e^{rt} = 0$
Characteristic equation: $r^2 + 2r + 1 = (r + 1)^2 = 0$.
The characteristic equation has a root $r = -1$ of multiplicity 2.
The complementary solution is therefore $y_c(t) = c_1e^{-t} + c_2te^{-t}$.

Let's assume a particular solution to the nonhomogeneous equation exists which looks like the function $g(t)$.

Assume $Y(t) = Ae^{-t}$ and we want to determine A. Note that A is a constant!
Substitute into the DE: $Y(t) = AE^{-t}$, $Y'(t) = -AE^{-t}$, $Y''(t) = AE^{-t}$,

$$\begin{aligned} Ae^{-t} + 2(-Ae^{-t}) + Ae^{-t} &= 2e^{-t} \\ 0 &= 2e^{-t} \end{aligned}$$

The method has failed! If our assumed solution was a solution, we could determine A so that the solution satisfied the nonhomogeneous equation. We can't do this. The method actually hasn't failed, what has happened is that we chose a form for the particular solution which is not a solution to the nonhomogeneous equation.

The problem is simple: our assumed solution is a solution of the homogeneous DE. Therefore, it cannot be a solution to the nonhomogeneous equation.

When you are using the method of undetermined coefficients, you **must** make sure that the assumed solution to the nonhomogeneous equation is not a solution of the associated homogeneous equation.

Another function which, when differentiated, has e^{-t} in the derivative is $Y(t) = Ate^{-t}$–which will also fail for the above reason.

If we assume that our solution looks like $Y(t) = At^2e^{-t}$, we no longer have something that appears in the complementary solution.

Differentiate and substitute:

$$\begin{aligned} y'' + 2y' + y &= 2e^{-t} \\ (2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}) + 2(2Ate^{-t} - At^2e^{-t}) + (At^2e^{-t}) &= 2e^{-t} \\ 2Ae^{-t} &= 2e^{-t} \\ \longrightarrow \quad A &= 1 \end{aligned}$$

The particular solution to the nonhomogeneous DE is $y_p(t) = t^2e^{-t}$.
The general solution is $y(t) = y_c(t) + y_p(t) = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$.

• Summary of how to use undetermined coefficients:

- If $g(t) = e^{\beta t}$, assume the particular solution is proportional to $e^{\beta t}$.
- If $g(t) = \sin\beta t, \cos\beta t$, assume the particular solution is proportional to a linear combination of $\sin\beta t$ and $\cos\beta t$.
- If $g(t)$ is a polynomial, than assume the particular solution is a polynomial of like degree.
- If $g(t)$ is a product of the above forms, assume the particular solution is the corresponding product.
- If $g(t)$ has more than one term, split the DE up and solve for a particular solution for each term individually.

When you have written down your assumed particular solution $Y(t)$, check to see if it has any overlap with the complementary solution $y_c(t)$. If it does, that part will satisfy the homogeneous DE and not the nonhomogeneous DE. Multiply that part of $Y(t)$ by t and check again.

Example (*Mathematica* online) Find the solution to the IVP $8y'' + 8y' + 2y = e^{-2t} + t\cos(2t)$, $y(0) = 0$ and $y'(0) = 0$.

Solve the homogeneous DE first: $8y'' + 8y' + 2y = 0$.

Linear, constant coefficient $\Rightarrow$ assume $y = e^{rt}$. $y' = re^{rt}$, $y'' = r^2e^{rt}$.

Substitute into the homogeneous DE to get the characteristic equation:

$$\begin{aligned} 8r^2 + 8r + 2r &= 0 \\ 4r^2 + 4r + 1 &= 0 \\ (2r+1)^2 &= 0 \text{ factor, or use the quadratic formula} \\ r_1 &= -1/2 \text{ of multiplicity } 2 \end{aligned}$$

A fundamental set of solutions is $y_1 = e^{-rt} = e^{-t/2}, y_2 = te^{rt} = te^{-t/2}$.

The complementary solution is $y_c(t) = c_1e^{-t/2} + c_2te^{-t/2}$.

Since the $g(t) = e^{-2t} + t\cos(2t)$ is made up of an exponential, cosine, and polynomial, use undetermined coefficients to get a particular solution.

Based on the form of $g(t)$, assume

$$
\begin{aligned}
Y(t) =& A_1 e^{-2t} + (A_2 t + A_3)(A_4 \cos(2t) + A_5 \sin(2t)) \quad \text{(uses the rules for each type of function)} \\
Y(t) =& A_1 e^{-2t} + A_3 A_4 \cos(2t) + A_2 A_4 t \cos(2t) + A_3 A_5 \sin(2t) + A_2 A_5 t \sin(2t) \quad \text{(multiply out)} \\
Y(t) =& B_1 e^{-2t} + B_2 \cos(2t) + B_3 t \cos(2t) + B_4 \sin(2t) + B_5 t \sin(2t) \quad \text{(define new constants)}
\end{aligned}
$$

The important thing here is that you can identify the different (linearly independent) functions that appear in the assumed solution. You should have a single constant in front of each. The five different functions we have in this case are e^{-2t}, $\cos(2t)$, $t\cos(2t)$, $\sin(2t)$, $t\sin(2t)$.

Is there any overlap with y_c? No. That means this $Y(t)$ will be a solution to the nonhomogeneous DE. If there was an overlap we would only multiply the terms that overlap by t, not the entire $Y(t)$.

Our goal is to figure out the values of the constants B. Use *Mathematica* to differentiate this and substitute into the nonhomogeneous DE. The result is

$$
e^{-2t}(1-18B_1)+(30B_2-8B_3-16B_4-32B_5+30B_3t-16B_5t+t)\cos(2t)+(16B_2+32B_3+30B_4-8B_5+16B_3t+30B_5t)\sin(2t)=0
$$

We want this to be true no matter what the value of t is. This happens if the coefficients of the different functions of t that appear are set to zero.

$$
\begin{aligned}
e^{-t}:& \quad 1 - 18B_1 = 0 \\
\cos(\pi t):& \quad 30B_2 - 8B_3 - 16B_4 - 32B_5 = 0 \\
t\cos(\pi t):& \quad 1 + 30B_3 - 16B_5 = 0 \\
\sin(\pi t):& \quad 16B_2 + 32B_3 + 30B_4 - 8B_5 = 0 \\
t\sin(\pi t):& \quad 16B_3 + 30B_5 = 0
\end{aligned}
$$

Note how the B_1 stands alone, and B_2–B_5 group together to give a set of 4 equations in 4 unknowns. This is because the $g(t)$ had two terms in it that we are treating all at the same time.

You can solve these equations using Cramer's rule or *Mathematica*.

$$
B_1 = \frac{1}{18} \quad B_2 = \frac{94}{4913} \quad B_3 = -\frac{15}{578} \quad B_4 = -\frac{104}{4913} \quad B_5 = \frac{4}{289}
$$

So a particular solution is known, and the general solution to the nonhomogeneous DE is given by

$$
y(t) = y_c + y_p = c_1 e^{-t/2} + c_2 t e^{-t/2} + \frac{1}{18}e^{-2t} + \frac{94}{4913}\cos(2t) - \frac{15}{578}t\cos(2t) + \frac{104}{4913}\sin(2t) + \frac{4}{289}t\sin(2t)
$$

Now that we have the general solution, we can use the initial conditions to determine the values of c_1 and c_2. Again, we can use Cramer's rule to do this by hand, or *Mathematica* to do the work for us.

We need to solve the two equations

$$
\begin{aligned}
y(0) = 0 &= \frac{6605}{88434} + c_1 \\
y'(0) = 0 &= -\frac{8377}{88434} - \frac{c_1}{2} + c_2
\end{aligned}
$$

which has the solution

$$
c_1 = -\frac{6605}{88434}, \qquad c_2 = \frac{199}{3468}.
$$

The solution to the IVP is

$$
y(t) = -\frac{6605}{88434}e^{-t/2} + \frac{199}{3468}te^{-t/2} + \frac{1}{18}e^{-2t} + \frac{94}{4913}\cos(2t) - \frac{15}{578}t\cos(2t) + \frac{104}{4913}\sin(2t) + \frac{4}{289}t\sin(2t).
$$

Example (on handout) Find the general solution of the DE $u'' + w_0^2 u = \cos(wt)$, where $w_0^2 \neq w^2$.

First, solve the associated homogeneous equation $u'' + w_0^2 u = 0$.
Assume $u = e^{rt}$.
Substitute into the DE: $r^2 e^{rt} + w_0^2 e^{rt} = 0$.
Characteristic equation: $r^2 + w_0^2 = 0$.
Roots of the characteristic equation are complex: $r = \pm w_0 = \lambda \pm \mu$.
Therefore, $\lambda = 0, \mu = w_0$.
A fundamental set of solutions is $u_1 = \cos w_0 t$, $u_2 = \sin w_0 t$.
The complementary solution is therefore $u_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Get a particular solution of the nonhomogeneous equation. Assume $U(t) = A\cos wt + B\sin wt$.
Substitute into the DE:

$$\begin{aligned}
u'' + w_0^2 u &= \cos(wt) \\
(-Aw^2\cos wt - Bw^2\sin wt) + w_0^2(A\cos wt + B\sin wt) &= \cos wt \\
A(w_0^2 - w^2)\cos wt + B(w_0^2 - w^2)\sin wt &= \cos wt \\
\Rightarrow \quad A(w_0^2 - w^2) = 1 & \qquad B(w_0^2 - w^2) = 0 \\
A = \frac{1}{w_0^2 - w^2} & \qquad B = 0
\end{aligned}$$

The particular solution of the nonhomogeneous DE is $y_p(t) = \dfrac{1}{w_0^2 - w^2}\cos wt$.

The general solution is $y(t) = y_c(t) + y_p(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \dfrac{1}{w_0^2 - w^2}\cos wt$.

Example (on handout) Find a particular solution of the DE $u'' + w_0^2 u = \cos(w_0 t)e^t$.

First, solve the associated homogeneous equation $u'' + w_0^2 u = 0$.
This is the same as in the previous problem.
The complementary solution is therefore $u_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Get a particular solution of the nonhomogeneous equation. Assume $U(t) = (\bar{A}\cos w_0 t + \bar{B}\sin w_0 t)(\bar{C}e^t) = (A\cos w_0 t + B\sin w_0 t)e^t$. This does not have overlap with the complementary solution, so it should be a solution of the nonhomogeneous equation.

Substitute into the DE; use *Mathematica* to help with derivatives. Details are online.

$$\begin{aligned}
e^t \sin w_0 t(B - 2Aw_0) + e^t \cos w_0 t(A + 2Bw_0) &= e^t \cos w_0 t \\
\Rightarrow \quad B - 2Aw_0 = 0 & \qquad A + 2Bw_0 = 1 \\
A = \frac{1}{1 + 4w_0^2} & \qquad B = \frac{2w_0}{1 + 4w_0^2}
\end{aligned}$$

The particular solution of the nonhomogeneous DE is $y_p(t) = \left(\dfrac{1}{1 + 4w_0^2}\cos w_0 t + \dfrac{2w_0}{1 + 4w_0^2}\sin w_0 t\right)e^t$.