Nonhomogeneous Equations of higher order: $L[y] = g(t)$.

2401 Differential Equations Section 4.3 Undetermined Coefficients[1]

The only thing to be aware of is that for higher order equations the characteristic equation may have roots of multiplicity greater than two, so be prepared to adjust you assumed solution accordingly (i.e. multiply by a large enough power of t to get rid of any proposed solutions which are solutions of the homogeneous equation).

The method is identical in form to that used for second order equations.

The algebra may be daunting to calculate the solutions; a computer algebra system can be a huge aid solution.

Example Find the solution to

$$y^{(4)} + 2y'' + y = 3\sin t - 5\cos t, y(0) = 0, y'(0) = 1, y''(0) = 2, y'''(0) = 3.$$

This solution is very well documented online; what appears here is a minimum explanation solution.

Assume $y = e^{rt}$, and we arrive at the characteristic equation: $r^4 + 2r^2 + 1 = 0$.
Roots of the characteristic equation are $r = \pm i$ of multiplicity 2.
A fundamental set of solutions is $y_1 = \cos t$, $y_2 = \sin t$, $y_3 = t\cos t$, $y_4 = t\sin t$.
The complementary solution is $y_c(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t$.

To get a particular solution we will use undetermined coefficients.
Assume a solution exists of the form: $Y(t) = A\cos t + B\sin t$.
This will fail since parts of our assumed solution appear in the complementary solution.
Therefore, assume instead that $Y(t) = At^2\cos t + Bt^2\sin t$.
Differentiate and substitute into the differential equation (use *Mathematica*):

$$-8(A\cos t + B\sin t) = 3\sin t - 5\cos t$$

Therefore, $-8A = -5, -8B = 3 \longrightarrow A = 5/8, B = -3/8$.
A particular solution is $y_p(t) = 5/8t^2\cos t - 3/8t^2\sin t$.
The general solution is $y(t) = y_c(t) + y_p(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t + 5/8t^2\cos t - 3/8t^2\sin t$.

Use the initial conditions to determine the constants. Taking derivatives and evaluating the initial conditions would be a lot of work to do by hand. This is definitely best done using *Mathematica* !

[1] Lecture is online, work mostly through *Mathematica* file which is also online. Variation of Parameters for $n \geq 3$ not on test.

The system of equations you arrive at is given by:

$$\begin{aligned} c1 &= 0 \\ c_2 + c_3 &= 1 \\ \frac{5}{4} - c_1 + 2c_4 &= 2 \\ -\frac{9}{4} - c_2 - 3c_3 &= 3 \end{aligned}$$

which has solution $c_1 = 0, c_2 = 33/8, c_3 = -25/8, c_4 = 3/8$.

The initial value problem has solution:

$$y(t) = \frac{5}{8}\cos(t)t^2 - \frac{3}{8}\sin(t)t^2 - \frac{25}{8}\cos(t)t + \frac{3}{8}\sin(t)t + \frac{33\sin(t)}{8}$$

2401 Differential Equations Section 4.4 Variation of Parameters

We use this when we can't use undetermined coefficients. Undetermined coefficients is much easier, so stick with it if possible!

Variation of Parameters in General

• Assume we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ for the homogeneous equation. Then the general complementary solution is

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

Assume a particular solution of the nonhomogeneous equation exists of the form:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t).$$

Since we have n functions u_n to determine, we shall have to specify n conditions. One condition is that $Y(t)$ satisfy the differential equation:

$$L[Y] = g(t).$$

The other conditions are chosen to simplify the calculation as much as possible. Everything is a function of time, so drop that notation:

$$Y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

• Now start taking derivatives:

First Condition: $u_1'y_1 + u_2'y_2 + \cdots + u_n'y_n = 0$

First Derivative: $Y' = u_1y_1' + u_2y_2' + \cdots + u_ny_n'$

Second Condition: $u_1'y_1' + u_2'y_2' + \cdots + u_n'y_n' = 0$

Second Derivative: $Y^{(2)} = u_1y_1^{(2)} + u_2y_2^{(2)} + \cdots + u_ny_n^{(2)}$

• Continue this procedure, to get the following:

$n-1$ conditions: $u_1'y_1^{(m)} + u_2'y_2^{(m)} + \cdots + u_n'y_n^{(m)} = 0, \quad m = 0, 1, 2, \ldots, n-2$

$n-1$ derivatives: $Y^{(m)} = u_1y_1^{(m)} + u_2y_2^{(m)} + \cdots + u_ny_n^{(m)}, \quad m = 0, 1, 2, \ldots, n-1$

The nth derivative is therefore: $Y^{(n)} = (u_1y_1^{(n)} + u_2y_2^{(n)} + \cdots + u_ny_n^{(n)}) + (u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)})$.

Substitute all this into the DE, collect terms, use $L[y_i] = 0$, and you will arrive at

$$u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = g$$

• This equation plus the $n-1$ conditions gives an algebraic system, n equations for the n unknowns u_i':

$$\begin{aligned}
&u_1'y_1^{(0)} + u_2'y_2^{(0)} + \cdots + u_n'y_n^{(0)} = 0\\
&u_1'y_1^{(1)} + u_2'y_2^{(1)} + \cdots + u_n'y_n^{(1)} = 0\\
&u_1'y_1^{(2)} + u_2'y_2^{(2)} + \cdots + u_n'y_n^{(2)} = 0\\
&\vdots\\
&u_1'y_1^{(n-2)} + u_2'y_2^{(n-2)} + \cdots + u_n'y_n^{(n-2)} = 0\\
&u_1'y_1^{(n-1)} + u_2'y_2^{(n-1)} + \cdots + u_n'y_n^{(n-1)} = g
\end{aligned}$$

The existence of a solution of this algebraic system is that $W(y_1, y_2, \ldots, y_n) \neq 0$, which have since the y_i form a fundamental set of solutions (linearly independent).

The solution to the system is found using **Cramer's Rule**:

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}, m = 1, 2, \ldots, n,$$

where $W(t) = W(y_1, y_2, \ldots, y_n)$ and $W_m(t)$ is found from $W(t)$ by replacing the mth column by the column $(0, 0, 0, \ldots, 1)^T$. This is because the right hand side of the system of equations are all zero except for the one that is $g(t)$, and we have factored out the $g(t)$ in the equation for $u_m'(t)$.

• A particular solution of the nonhomogeneous equation is therefore given by

$$y_p(t) = Y(t) = \sum_{m=1}^{n} Y_i(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds.$$

Since we have already shown how to implement this method from first principles for $n = 2$, if $n \geq 3$ we can use this formula directly.

Note that on any Test I want you to be able to explain variation of parameters in detail for $n = 2$ by working through the details for a particular problem. No formulas! There will be no problems on a Test involving variation of parameters for $n \geq 3$. On Assignments, you may use the formulas for variation of parameters if $n \geq 3$.

Example Find the solution to $y''' - y' = \cos^2 t$ by variation of parameters.

Mathematica is again a big help in solving this differential equation. The *Mathematica* detailed solution is online.

First, solve the homogeneous equation for the complementary solution.
Assume $y(t) = e^{rt}$.
Characteristic equation: $r^3 - r = r(r-1)(r+1) = 0$.
Roots are $r_1 = 0, r_2 = +1, r_2 = -1$.
A fundamental set of solutions is $y_1 = 1, y_2 = e^t, y_3 = e^{-t}$.
The complementary solution is therefore: $y_c(t) = c_1 + c_2e^t + c^3e^{-t}$.

The Wronskain of the fundamental set of solutions is:

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2$$

Get a particular solution of the form $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$.
Note that the nonhomogeneous term is $g(t) = \cos^2 t$.

$\underline{Y_1(t)}$:

$$W_1(y_1, y_2, y_3)(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2$$

Since t_0 is arbitrary, let's set it equal to zero, and we have:

$$\int_{t_0}^{t} \frac{g(s)W_1(s)}{W(s)} ds = \int_0^t \frac{(\cos^2 s)(-2)}{2} ds = \frac{1}{2}(-t - \cos t \sin t)$$

$$Y_1(t) = y_1 \int_{t_0}^{t} \frac{g(s)W_1(s)}{W(s)} ds = \frac{1}{2}(-t - \cos t \sin t).$$

$\underline{Y_2(t)}$:

$$W_2(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t}$$

$$\int_{t_0}^{t} \frac{g(s)W_2(s)}{W(s)} ds = \int_{0}^{t} \frac{(\cos^2 s)e^{-s}}{2} ds = \frac{1}{20}\left(6 - e^{-t}(\cos(2t) - 2\sin(2t) + 5)\right)$$

$$Y_2(t) = y_2 \int_{t_0}^{t} \frac{g(s)W_2(s)}{W(s)} ds = \frac{1}{20}\left(6e^{t} - \cos(2t) + 2\sin(2t) - 5\right)$$

<u>$Y_3(t)$:</u>

$$W_3(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t$$

$$\int_{t_0}^{t} \frac{g(s)W_3(s)}{W(s)} ds = \int_{0}^{t} \frac{(\cos^2 s)e^{s}}{2} ds = \frac{1}{20}\left(e^{t}(\cos(2t) + 2\sin(2t) + 5) - 6\right)$$

$$Y_3(t) = y_3 \int_{t_0}^{t} \frac{g(s)W_3(s)}{W(s)} ds = \frac{1}{20}\left(\cos(2t) + 2\sin(2t) + 5 - 6e^{-t}\right)$$

The complete particular solution is given by

$$y_p(t) = Y(t) = Y_1(t) + Y_2(t) + Y_3(t) = \frac{1}{10}\left(-5t - 3e^{-t} + 3e^{t} - \cos t \sin t\right)$$

The terms involving e^t and e^{-t} are part of the complementary solution and so can be dropped from the particular solution, although simply using the above as the particular solution would also be correct.

The particular solution we choose will be

$$y_p(t) = \frac{1}{10}\left(-5t - \cos t \sin t\right)$$

The general solution to the nonhomogeneous differential equation is

$$y(t) = y_c(t) + y_p(t) = c_1 + c_2 e^{t} + c_3 e^{-t} - \frac{t}{2} - \frac{1}{10}\cos t \sin t$$