Section 5.2 Series Solution Near Ordinary Point

We are interested in second order homogeneous linear differential equations with variable coefficients.

- Consider the differential equation:

  $$P(x)y'' + Q(x)y' + R(x)y = 0.$$

- To simplify our lives, we will consider P, Q, R to be polynomials, although this method works for any analytic functions P, Q, R.
- If the function is not a polynomial but is analytic, for example $P(x) = \sin x$, we must replace $P(x)$ with its Taylor series.
- Suppose we wish to find a solution near $x = x_0$. The solution in an interval I about x_0 is closely dependent on the behaviour of P in that region.
- If x_0 is such that $P(x_0) \neq 0$, then since P is continuous, there is an interval around x_0 where $P(x) \neq 0$, and so we can divide by $P(x)$ and then look for a solution in that interval:

  $$y'' + p(x)y' + q(x)y = 0$$

- According to the existence and uniqueness theorems, there exists a unique solution to this differential equation in the interval that satisfies the initial conditions $y(x_0) = y_0, y'(x_0) = {y'}_0$.
- Finding these solutions can be difficult, and the difficulty depends on what point you assume a series solution is about.

Classifying Points (from 5.4)

For the differential equation $y'' + p(x)y' + q(x)y = 0$, we classify the point x_0 in the following manner.

1. If $p(x)$ and $q(x)$ are analytic at x_0 (remember, analytic at x_0 means the functions have a Taylor series with nonzero radius of convergence), then the point x_0 is called an ordinary point of the differential equation.
2. If x_0 is not an ordinary point, it is a singular point.
3. If x_0 is a singular point and $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at x_0, then x_0 is a regular singular point.
4. If x_0 is singular but not regular, it is called an irregular singular point.

The type of series solution you assume depends on the classification of the point you are expanding about.

We will look at solutions about ordinary points and regular singular points.

Solutions about irregular singular points require more advanced techniques (asymptotics and dominant balance), and there is no comprehensive theory for finding the solution.

Advanced Mathematical Methods for Scientists and Engineers (Asymptotic Methods and Perturbation Theory), Bender and Orszag. It was reprinted in 1999, and I think it is a wonderful book.

You can also classify the point $x_0 = \infty$ by making the transformation $x = 1/t$. This is called finding singularities at infinity.

Example Consider the differential equation $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$. Find and classify all the singular points.

Identify

$$p(x) = \frac{Q(x)}{P(x)} = \frac{3(x-1)}{(x+2)^2(x-1)} = \frac{3}{(x+2)^2}$$

which has a singular point at $x_0 = -2$, and

$$q(x) = \frac{R(x)}{P(x)} = \frac{-2(x+2)}{(x+2)^2(x-1)} = \frac{-2}{(x+2)(x-1)}$$

which has singular points at $x_0 = -2, +1$.

The singular points are $x_0 = -2, +1$.

Classify: $x_0 = -2$:

$$(x - x_0)p(x) = (x+2)\frac{3}{(x+2)^2} = \frac{3}{x+2}$$

$$(x - x_0)^2 q(x) = (x+2)^2 \frac{-2}{(x+2)(x-1)} = \frac{-2(x+2)}{x-1}$$

The second has a convergent Taylor series about $x_0 = -2$, but the first does not. Therefore $x_0 = -2$ is an irregular singular point.

Classify: $x_0 = +1$:

$$(x - x_0)p(x) = (x-1)\frac{3}{(x+2)^2} = \frac{3(x-1)}{(x+2)^2}$$

$$(x - x_0)^2 q(x) = (x-1)^2 \frac{-2}{(x+2)(x-1)} = \frac{-2(x-1)}{(x+2)}$$

These both have convergent Taylor series for some nonzero interval about $x_0 = +1$, so $x_0 = +1$ is a regular singular point.

All other points are ordinary points.

Series Solutions around an Ordinary Point x_0

If you want a series solution about an ordinary point x_0, assume a solution exists which is of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad |x - x_0| < \rho \neq 0.$$

We are guaranteed to find two solutions that form a fundamental set of solutions.

Procedure:

1. Differentiate and substitute into DE.
2. simplify the DE using the techniques for combining power series we discussed in 5.1, and determine the recursion relations for the coefficients a_n.
3. Expand out the recursion relations to see if you can determine the pattern.
4. Use *Mathematica* to determine the underlying functions in the fundamental set of solutions.

- A truncated power series only applies in a local area about the center of expansion ($x = 1$ here).
- The idea that fundamental functions are defined in terms of differential equations is a powerful one.
- In this manner polynomials like the Hermite, Legendre, Laguerre, Airy functions, Bessel functions, and others are obtained.

Example of Process Find a series solution of $y'' + y = 0$ about $x_0 = 1$.

Since $p(x) = 0$ and $q(x) = 1$ are analytic at $x_0 = 1$, the point $x_0 = 0$ is an ordinary point.

The function $p(x)$ and $q(x)$ are already expressed as Taylor series about $x_0 = 1$, if they weren't we would need to replace them by their respective Taylor series expansions.

Therefore, assume a solution of the form: $y = \displaystyle\sum_{n=0}^{\infty} a_n (x-1)^n, \quad |x| < \rho.$

Differentiate:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
y' &= \sum_{n=0}^{\infty} n a_n (x-1)^{n-1} \\
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2}
\end{aligned}$$

Substitute into the DE:

$$\sum_{n=0}^{\infty} n(n+1) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] (x-1)^n = 0$$

Each coefficient must be zero for the series to be zero, and this gives us the <u>recurrence relation</u>:

$$(n+2)(n+1) a_{n+2} + a_n = 0, \qquad n = 0, 1, 2, 3, \ldots$$

The a_0 and a_1 are not determined by the recurrence relation, so they are arbitrary, and the recurrence relation yields:

$$\begin{aligned}
a_{n+2} &= \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3, ... \\
a_0 &= \text{arbitrary, not equal to zero} \\
a_1 &= \text{arbitrary, not equal to zero} \\
a_2 &= -\frac{a_0}{1 \cdot 2} = -\frac{a_0}{2!} \\
a_3 &= -\frac{a_1}{2 \cdot 3} = -\frac{a_1}{3!} \\
a_4 &= -\frac{a_2}{3 \cdot 4} = +\frac{a_0}{4!} \\
a_5 &= -\frac{a_3}{4 \cdot 5} = +\frac{a_1}{5!} \\
a_6 &= -\frac{a_4}{5 \cdot 6} = -\frac{a_0}{6!} \\
a_7 &= -\frac{a_5}{6 \cdot 7} = -\frac{a_1}{7!}
\end{aligned}$$

The pattern is easy to find here, since the original DE is one we could solve without using series solutions (it was constant coefficient). The pattern is different if we have even or odd powers of x (the even have an a_0, the odd a_1), so we should write

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{k=0}^{\infty} a_{2k}(x-1)^{2k} + \sum_{k=0}^{\infty} a_{2k+1}(x-1)^{2k+1} \\
a_{2k} &= (-1)^k \frac{1}{(2k)!} a_0 \qquad \text{(find a pattern, write in closed form)} \\
a_{2k+1} &= (-1)^k \frac{1}{(2k+1)!} a_1 \\
y &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x-1)^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x-1)^{2k+1} \\
&= a_0 \cos(x-1) + a_1 \sin(x-1) \qquad \text{(identify the function the series represents)}
\end{aligned}$$

The a_0, a_1 are the arbitrary constants of the problem (what we have been calling c_1, c_2 usually).

The fundamental set of solution is given by:

$$\begin{aligned}
y_1(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x-1)^{2k} = \cos(x-1) \\
y_2(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x-1)^{2k+1} = \sin(x-1)
\end{aligned}$$

In this case we were able to

1. find a general closed from for the two solutions,
2. identify the closed form as the trig functions sine and cosine.

In general, neither of these things may be possible! Frequently, we can identify the pattern, but not what the function is that the sum represents.

Since we identified the underlying function, we do not have to worry about where the series converges.