Section 5.3 Series Solution Near Ordinary Point

Example Find a series solution around $x = x_0 = 0$ to the differential equation $(1-x)y'' + xy' - y = 0$.

The coefficients in the differential equation are:

$$\begin{aligned} P(x) &= 1-x, \\ Q(x) &= x, \\ R(x) &= -1. \end{aligned}$$

These are already in powers of $(x-0)$, so we do not need to do any Taylor series expansions of the coefficients.

Since $p(x) = \dfrac{x}{1-x}$ and $q(x) = \dfrac{-1}{1-x}$ are analytic at $x_0 = 0$ (basically, the functions are not infinite at x_0), $x_0 = 0$ is an ordinary point.

Assume a solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < \rho.$$

Differentiate:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}, \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \end{aligned}$$

Substitute into the differential equation:

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 1: Push the P, Q, R into the sums:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Step 2: Get the same power of x in each term. Replace $m = n+1$ in the two middle sums, other sums let $m = n$:

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty}(m+1)m a_{m+1}x^m + \sum_{m=1}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

Step 3: Get all the summations starting at the same point. Generally, choose the highest and make all the summations start there. In this case we take out the $m = 0$ terms of the first and last terms:

$$2 \cdot 1\, a_2 x^0 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty}(m+1)m a_{m+1}x^m + \sum_{m=1}^{\infty} m a_m x^m - a_0 x^0 - \sum_{m=1}^{\infty} a_m x^m = 0$$

Now collect all the terms together:

$$(2a_2 - a_0)x^0 + \sum_{m=1}^{\infty}[(m+2)(m+1)a_{m+2} - (m+1)ma_{m+1} + ma_m - a_m]x^m = 0$$

We set the coefficients of x equal to zero (since the entire series equals zero). This is really equating powers of x, so keep that in mind if you are equating two series!

$$\begin{aligned}
&2a_2 - a_0 = 0, \quad n = 0, \\
&(m+2)(m+1)a_{m+2} - (m+1)ma_{m+1} + ma_m - a_m = 0, \qquad m = 1, 2, 3, \ldots.
\end{aligned}$$

Notice that if we take $m = 0$ in the second relation, we get $2a_2 - a_0 = 0$, so we can combine these two relations. This is not always going to happen! The recurrence relation is therefore:

$$(m+2)(m+1)a_{m+2} - (m+1)ma_{m+1} + (m-1)a_m = 0, \quad m = 0, 1, 2, 3, \ldots$$

Now, use the recurrence relation to determine the coefficients a_n. $n = 0$ specifies a_2 in terms of a_1 and a_0. Hence, a_0 and a_1 are arbitrary. They represent the constants of integration.

$$\begin{aligned}
a_{m+2} &= \frac{(m+1)m\, a_{m+1} - (m-1)a_m}{(m+2)(m+1)}, \qquad m = 0, 1, 2, 3 \ldots \\
a_0 &= \text{arbitrary} \\
a_1 &= \text{arbitrary} \\
a_2 &= \frac{a_0}{2!} \\
a_3 &= \frac{2a_2}{3 \cdot 2} = \frac{2}{3 \cdot 2}\frac{a_0}{2!} = \frac{a_0}{3!} \\
a_4 &= \frac{3 \cdot 2\, a_3 - a_2}{4 \cdot 3} = \frac{3 \cdot 2 \frac{a_0}{3!} - \frac{a_0}{2!}}{4 \cdot 3} = \frac{a_0}{4!} \\
a_5 &= \frac{4 \cdot 3\, a_4 - 2\, a_3}{5 \cdot 4} = \frac{4 \cdot 3 \frac{a_0}{4!} - 2\frac{a_0}{3!}}{5 \cdot 4} = \frac{a_0}{5!}
\end{aligned}$$

In general, we have

$$a_m = \frac{a_0}{m!}, \qquad m = 0, 2, 3, 4, \ldots$$

Note that the $m = 1$ term is not included in the above. The solution to the differential equation is:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{m=0}^{\infty} a_m x^m \\
&= a_0 + a_1 x + \sum_{m=2}^{\infty} \frac{a_0}{m!} x^m \\
&= a_0 \left[1 + \sum_{m=2}^{\infty} \frac{1}{m!} x^m\right] + a_1 x \\
&= a_0 y_1(x) + a_1 y_2(x)
\end{aligned}$$

The constants of integration are a_0, a_1, which we previously called c_1, c_2. A fundamental set of solutions is $\{y_1, y_2\}$,

$$y_1 = 1 + \sum_{m=2}^{\infty} \frac{1}{m!} x^m, \qquad y_2 = x.$$

The solution y_2 is obviously convergent for all x.

In this case we can identify the infinite series as part of the Taylor series expansion for e^x, so we have

$$y_1 = 1 + \sum_{m=2}^{\infty} \frac{1}{m!} x^m = 1 - 1 - x + \sum_{m=0}^{\infty} \frac{1}{m!} x^m = -x + e^x.$$

Since we were able to identify the underlying function the series represents, we do not need to worry about radius of convergence of our series.

If we did need to consider radius of convergence of series, there is a helpful theorem.

Theorem 5.3.1 *If x_0 is an ordinary point of the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution is*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2,$$

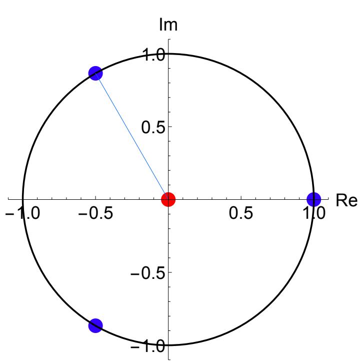
where a_0 and a_1 are arbitrary, and the y_1 and y_2 are linearly independent series solutions that are analytic at x_0. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

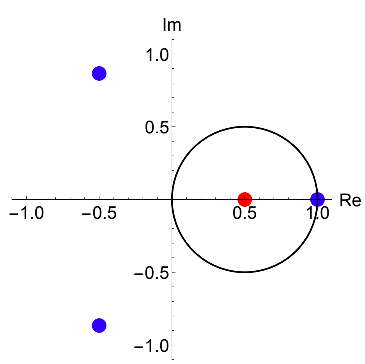
Example Determine the minimum radii of convergence for the series solutions about $x_0 = 0$ and $x_0 = 1/2$ for the differential equation $(1 - x^3)y'' + 4xy' + y = 0$.

We do not have to find the series solutions to answer this question. Identify $p(x) = \dfrac{4x}{1 - x^3}$ and $q(x) = \dfrac{1}{1 - x^3}$.

The complex poles of p and q are when $1 - x^3 = 0$, or $x = 1^{1/3}$ which are the third roots of unity. Therefore, $x = 1, -1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2$. Both $x_0 = 0$ and $x_0 = 1/2$ are ordinary points, so Theorem 5.3.1 applies.

We have the following diagrams for $x_0 = 0$ (left) and $x_0 = 1/2$ (right) (red points are x_0, blue points are the complex poles of p and q):





For $x_0 = 0$: The distance to nearest complex pole is 1. Therefore, the minimum radius of convergence of the series solution about $x_0 = 0$ is $\rho = 1$.

For $x_0 = 1/2$: The distance to nearest complex pole is 1/2. Therefore, the minimum radius of convergence of the series solution about $x_0 = 1/2$ is $\rho = 1/2$.