2401 Differential Equations Section Chapter 5.6 Series Solution

Here we will conclude our look at series solutions by looking at series solutions in general.

What we see here is verification of the form of the solutions we saw earlier, and although we should not try to use these structure to solve the differential equations by hand, the relations we will find are well suited to computer implementation.

We will need the series multiplication result: $\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$

We are considering the differential equation of the form $y'' + p(x)y' + q(x)y = 0$.

Series Solution about $x_0 = 0$, an Ordinary Point: General Case

If the point $x_0 = 0$ is an ordinary point, then the p and q are analytic, which means they have a Taylor series about $x_0 = 0$ with a nonzero radius of convergence. Therefore, we can write them as

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Assume:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y' &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} p_n x^n \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} q_n x^n \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k (n-k+1) a_{n-k+1} \right) x^n + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} q_k a_{n-k} \right) x^n &= 0 \\ \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} p_k (n-k+1) a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} \right) x^n &= 0 \end{aligned}$$

For this to be zero for all values of x, each coefficient of x must be zero:

$$(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} \Big(p_k (n-k+1) a_{n-k+1} + q_k a_{n-k} \Big) = 0, \quad n = 0, 1, 2, 3, \ldots \tag{1}$$

Recursion Relations

We will always be able to solve Eq. (1) recursively, and get two solutions in terms of the two unspecified constants a_0 and a_1. The solution will split up into two infinite series, one with a_0 factors, the other with a_1 factors. This happens because the differential equation is linear.

Notice that in general, the coefficient a_{n+2} depends on all the previous coefficients.

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} \sum_{k=0}^{n} \Big(p_k (n-k+1) a_{n-k+1} + q_k a_{n-k} \Big), \quad n = 0, 1, 2, 3, \ldots$$

This can be implemented on a computer in the following manner:

$$\begin{aligned} a_0 &= c_1 \\ a_1 &= c_2 \\ a_n &= -\frac{1}{n(n-1)} \sum_{k=0}^{n-2} \Big(p_k (n-k-1) a_{n-k-1} + q_k a_{n-k-2} \Big), \quad n = 2, 3, \ldots \end{aligned}$$

Series Solution about $x_0 = 0$, a Regular Singular Point: General Case

If the point $x_0 = 0$ is a regular singular point, then $xp(x)$ and $x^2 q(x)$ are analytic, which means they have a Taylor series about $x_0 = 0$ with a nonzero radius of convergence. Therefore, we can write them as

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Assume:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}. \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned}
y'' + p(x)y' + q(x)y &= 0 \\
x^2y'' + x\left(xp(x)\right)y' + x^2q(x)y &= 0 \\
x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + x\left(\sum_{n=0}^{\infty}p_nx^n\right)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + \left(\sum_{n=0}^{\infty}q_nx^n\right)\sum_{n=0}^{\infty}a_nx^{n+r} &= 0 \\
\cancel{x^r}\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^n + \left(\sum_{n=0}^{\infty}p_nx^n\right)\sum_{n=0}^{\infty}(n+r)a_nx^n + \left(\sum_{n=0}^{\infty}q_nx^n\right)\sum_{n=0}^{\infty}a_nx^n\right) &= 0 \\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^n + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}p_k(n-k+r)a_{n-k}\right)x^n + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}q_ka_{n-k}\right)x^n &= 0 \\
\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)a_n + \sum_{k=0}^{n}p_k(n-k+r)a_{n-k} + \sum_{k=0}^{n}q_ka_{n-k}\right]x^n &= 0
\end{aligned}$$

For this to be zero for all values of x, each coefficient of x must be zero:

$$(n+r)(n+r-1)a_n + \sum_{k=0}^{n}\left(p_k(n-k+r)a_{n-k} + q_ka_{n-k}\right) = 0, \quad n = 0, 1, 2, 3, \ldots \tag{2}$$

Indicial and Recursion Relations

We need to get an indicial equation and recurrence relations from these; if we take the $n = 0$ part to be the indicial equation:

$$F(r) = r(r-1) + rp_0 + q_0 = 0$$

with the condition that $a_0 \neq 0$, which is the arbitrary constant in the solution.

Notice the indicial equation is the indicial equation for the associated Euler equation, where the expansions for $xp(x) \sim p_0$ and $x^2q(x) \sim q_0$ are replaced by the constant term. The roots of the indicial equation are r_1 and r_2.

Then the recurrence relations are:

$$(n+r)(n+r-1)a_n + \sum_{k=0}^{n}\left(p_k(n-k+r)a_{n-k} + q_ka_{n-k}\right) = 0, \quad n = 1, 2, 3, \ldots$$

We can't quite work with this form–notice that $k = 0$ in the sum gives us an a_n, which is what we want to solve for. Therefore, we have to take the $k = 0$ part out of the sum, and then solve for a_n, which gives us

$$\begin{aligned}
a_n &= -\frac{1}{(n+r)(n+r-1) + p_0(n+r) + q_0}\sum_{k=1}^{n}\left(p_k(n-k+r) + q_k\right)a_{n-k}, \quad n = 1, 2, 3, \ldots \\
&= -\frac{1}{F(n+r)}\sum_{k=1}^{n}\left(p_k(n-k+r) + q_k\right)a_{n-k}, \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Notice that in general, the coefficient a_n depends on all the previous coefficients.

If $r_{1,2}$ are complex, we will have no problems with division by zero, and we will obtain two complex valued solutions for which $r_{1,2} = \lambda \pm \mu i$. We can get real valued solutions by taking the real and imaginary parts of the solution as we have done before.

Real Roots: Assume $r_1 \geq r_2$. We can always get one solution to the differential equation using the larger of the two roots of the indicial equation, r_1.

For the root r_1, the recurrence relations are well defined, and everything works well.

$$a_0 = 1$$
$$a_n = -\frac{1}{F(n+r_1)} \sum_{k=1}^{n} \Big(p_k(n-k+r_1) + q_k \Big) a_{n-k}, \quad n = 1, 2, 3, \ldots$$

We would proceed to find a second solution using the root r_2, using the recurrence relations

$$a_0 = 1$$
$$a_n = -\frac{1}{F(n+r_2)} \sum_{k=1}^{n} \Big(p_k(n-k+r_2) + q_k \Big) a_{n-k}, \quad n = 1, 2, 3, \ldots$$

The only difficulty could be if $F(n+r_2) = 0$, which happens if $n + r_2 = r_1$, since r_1 and r_2 are the only solutions to $F(r) = 0$.

Therefore, if $r_1 - r_2 = n$, we get division by zero and we can't find a second solution from the recursion relation.

If the two roots are real and do not differ by an integer, a second solution can be found using the recurrence relation.

If the two roots are real and differ by an integer, a second solution can be found using reduction of order. The formula we worked out earlier for the second solution was:

$$y_2(x) = y_1(x) \int \frac{\exp(-\int p(x)dx)}{y_1^2(x)} dx.$$

Note that this involves inverting and squaring a power series!

There are other ways of getting the second solution–see the text.

The second solution may involve a logarithm for real roots that differ by an integer, and it is not unusual for the second solution to diverge at $x = 0$.

Real Repeated Roots: You can also use reduction of order to get a second solution for repeated roots.

Convergence: Consider the series alone, with out the $x^{r_{1,2}}$ part. As before, these series will converge at least with the radius of convergence of the minimum of the $xp(x)$ and $x^2q(x)$ radius of convergence. These functions are analytic at $x_0 = 0$. The singular behaviour, if any, is entirely contained in the $x^{r_{1,2}}$ factor!

If you can determine the underlying function the series represent, you need not worry about the radius of convergence of the series since you have the function and can use its domain to guide you.

To go to negative x, we end up with the same equations, so we can replace $x \to |x|$ and consider all $x \neq 0$.