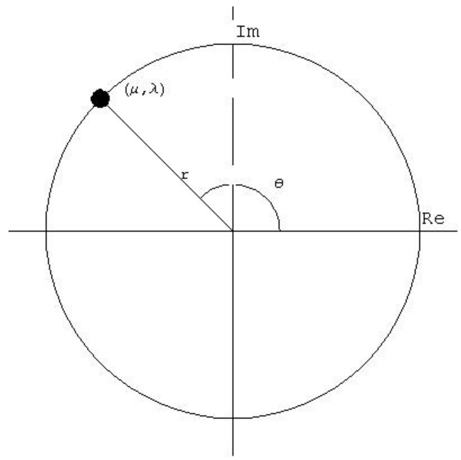
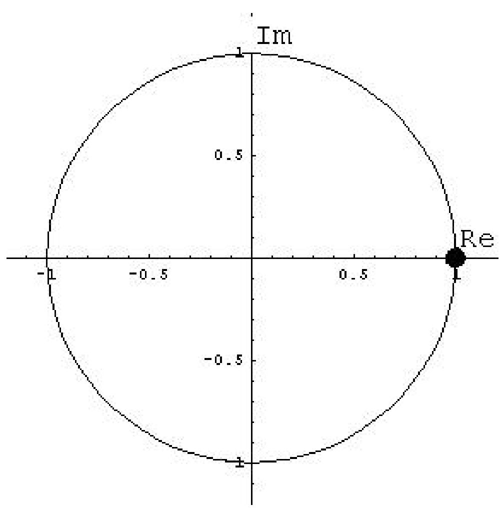
Math 2401: Roots of Unity

Here we will outline a procedure to find the roots of unity. The calculations will get progressively more complicated. The basic idea is to use the relation $1 = e^{2i\pi k}$ where k is an integer. The method can be expanded to other numbers using the relation $\mu + i\lambda = re^{i(\theta+2\pi k)}$, where:



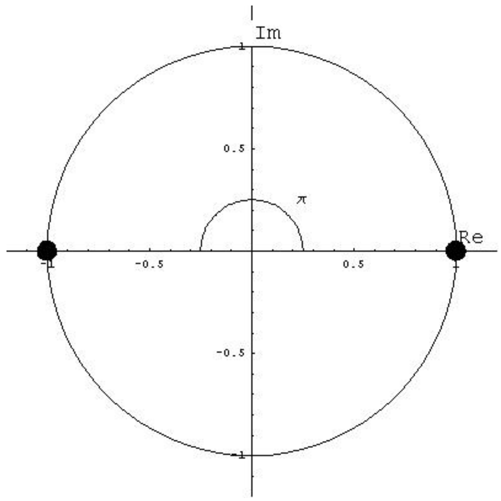
First root of unity:

$$1^{1/1} = 1$$



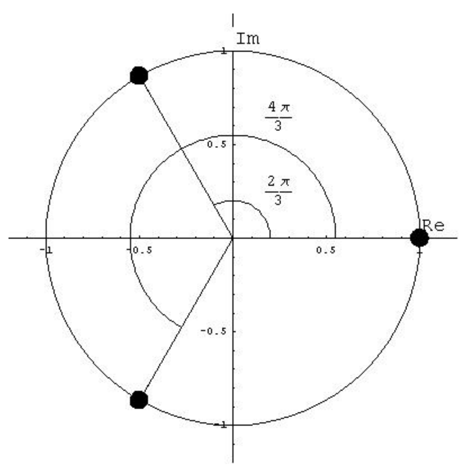
Second roots of unity:

$$\begin{aligned} 1^{1/2} &= \left(e^{i(2\pi k)}\right)^{1/2}, \quad k = 0, 1 \\ &= e^{i(\pi k)}, \quad k = 0, 1 \\ &= e^{0}, \text{ or } e^{i\pi} \\ &= 1, \text{ or } \cos\pi + i\sin\pi \\ &= 1, \text{ or } -1 \end{aligned}$$



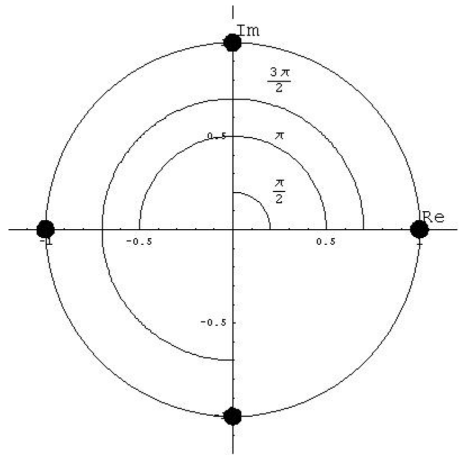
Third roots of unity:

$$
\begin{aligned}
1^{1/3} &= \left(e^{i(2\pi k)}\right)^{1/3}, \quad k = 0,1,2 \\
&= e^{i(2\pi k)/3}, \quad k = 0,1,2 \\
&= e^{0}, \text{ or } e^{i2\pi/3}, \text{ or } e^{i4\pi/3} \\
&= 1, \text{ or } \cos 2\pi/3 + i \sin 2\pi/3, \text{ or } \cos 4\pi/3 + i \sin 4\pi/3 \\
&= 1, \text{ or } -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \text{ or } -\frac{1}{2} - \frac{\sqrt{3}}{2} i
\end{aligned}
$$



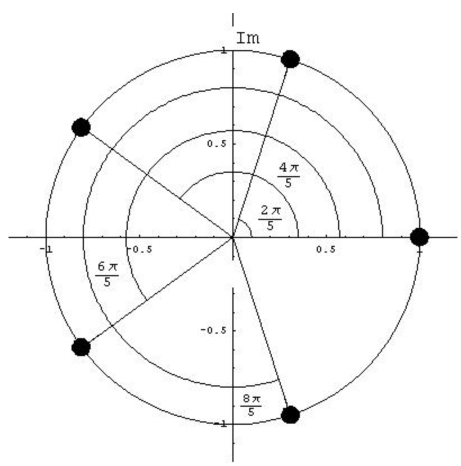
Fourth roots of unity:

$$
\begin{aligned}
1^{1/4} &= \left(e^{i(2\pi k)}\right)^{1/4}, \quad k = 0,1,2,3 \\
&= e^{i(\pi k)/2}, \quad k = 0,1,2,3 \\
&= e^{0}, \text{ or } e^{i\pi/2}, \text{ or } e^{i\pi}, \text{ or } e^{i3\pi/2} \\
&= 1, \text{ or } \cos \pi/2 + i \sin \pi/2, \text{ or } \cos \pi + i \sin \pi, \text{ or } \cos 3\pi/2 + i \sin 3\pi/2 \\
&= 1, \text{ or } i, \text{ or } -1, \text{ or } -i
\end{aligned}
$$



Fifth roots of unity:

$$\begin{aligned}
1^{1/5} &= \left(e^{i(2\pi k)}\right)^{1/5}, \quad k=0,1,2,3,4 \\
&= e^{i(2\pi k)/5}, \quad k=0,1,2,3,4 \\
&= e^0, \text{ or } e^{i2\pi/5}, \text{ or } e^{i4\pi/5}, \text{ or } e^{i6\pi/5}, \text{ or } e^{i8\pi/5} \\
&= 1, \text{ or } \cos 2\pi/5 + i\sin 2\pi/5, \text{ or } \cos 4\pi/5 + i\sin 4\pi/5, \\
&\quad \text{ or } \cos 6\pi/5 + i\sin 6\pi/5, \text{ or } \cos 8\pi/5 + i\sin 8\pi/5 \\
&= 1, \text{ or } \frac{1}{4}(-1+\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}, \text{ or } \frac{1}{4}(-1-\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})}, \\
&\quad \text{ or } \frac{1}{4}(-1-\sqrt{5}) - \frac{i}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})}, \text{ or } \frac{1}{4}(-1+\sqrt{5}) - \frac{i}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})} \\
&= 1, \text{ or } 0.309+0.951i, \text{ or } -0.809+0.588i, \text{ or } -0.809-0.588i, \text{ or } 0.309-0.951i
\end{aligned}$$



Note that in all cases the roots are equally spaced around the unit circle.

Application to Differential Equations

Now, we can use this information to solve the differential equation

$$y^{(5)} - y = 0.$$

If we assume the solution is $y = e^{rt}$, and substitute into the differential equation we arrive at the characteristic equation

$$\begin{aligned} y^{(5)} - y &= 0 \\ r^5 e^{rt} - e^{rt} &= 0 \\ r^5 - 1 &= 0 \quad \text{characteristic equation} \\ r &= 1^{1/5} \end{aligned}$$

We solved this above, and saw that one root is real, and the other four roots are complex conjugates. We have

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \cos 2\pi/5 + i \sin 2\pi/5 = \frac{1}{4}(-1+\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})} \\ r_3 &= \cos 4\pi/5 + i \sin 4\pi/5 = \frac{1}{4}(-1-\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})} \\ r_4 &= \cos 6\pi/5 + i \sin 6\pi/5 = \frac{1}{4}(-1-\sqrt{5}) - \frac{i}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})} \\ r_5 &= \cos 8\pi/5 + i \sin 8\pi/5 = \frac{1}{4}(-1+\sqrt{5}) - \frac{i}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})} \end{aligned}$$

The complex roots lead to complex valued solutions. These can be replaced by real valued solutions.

The complex conjugate pair r_2 and r_5 have $\lambda = \frac{1}{4}(-1+\sqrt{5})$, $\mu = \frac{1}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})}$.

The complex conjugate pair r_3 and r_4 have $\lambda = \frac{1}{4}(-1-\sqrt{5})$, $\mu = \frac{1}{2}\sqrt{\frac{1}{2}(5-\sqrt{5})}$.

The general solution is therefore (using $2\pi/5$ and $4\pi/5$ for notational convenience):

$$\begin{aligned} y(t) &= c_1 e^t + e^{t(\cos 2\pi/5)}\Big(c_2 \cos(t(\sin 2\pi/5)) + c_3 \sin(t(\sin 2\pi/5))\Big) \\ &\quad + e^{t(\cos 4\pi/5)}\Big(c_4 \cos(t(\sin 4\pi/5)) + c_5 \sin(t(\sin 4\pi/5))\Big) \end{aligned}$$

or

$$y(t) = c_1 e^t + e^{0.309t}\Big(c_2 \cos 0.951t + c_3 \sin 0.951t\Big) + e^{-0.809t}\Big(c_4 \cos 0.588t + c_5 \sin 0.588t\Big).$$