The midterm will include 5 questions:

- Part A will be 10 True/False.
- Part B is long answer, and you will choose four questions to answer out of seven provided.

You will have 65 minutes to complete the midterm.
You may use calculators, but you shouldn't need one to answer the questions.
To do well on the midterm you will need to have a strong mastery of the techniques–you will not have time to go down wrong paths and still complete the test.

Problem 1. Answer as True or False:

A nonlinear differential equation always has a general solution . . . . T F

A nonlinear differential equation always has a unique solution . . . . T F

An autonomous differential equation is of the form $dy/dt = f(y)$ and can be nonlinear in y . . . . T F

In the method of successive approximations (sometimes called Picard's iteration method), each member of the sequence of functions $\{\phi_n\}$ which is constructed satisfies the initial condition, but in general none will satisfy the differential equation . . . . T F

The differential equation $yy'' = \cos t$ is second order and linear in y . . . . T F

An autonomous differential equation will always have a general solution . . . . T F

The possible points of discontinuity, or singularities, can be identified without solving the problem for a linear differential equation . . . . T F

If $y_1(t)$ and $y_2(t)$ are two solutions of a differential equation, then $c_1y_1(t) + c_2y_2(t)$, where c_1, c_2 are constants, will also be a solution of the differential equation . . . . T F

If the characteristic equation associated with a differential equation has a real root r of multiplicity two, then a fundamental set of solutions is $y_1(t) = e^{rt}$ and $y_2(t) = re^{rt}$ . . . . T F

The Wronskian of $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ is $W(y_1, y_2)(t) = -e^{-5t}$ . . . . T F

If the Wronskian of two differentiable functions is not equal to zero on some interval I, then the two functions are linearly dependent on I . . . . T F

A linear, second order differential equation can have more than one fundamental set of solutions . . . . T F

The Wronskian of $y_1(t) = \sin^2 t$ and $y_2(t) = \cos^2 t$ is $W(y_1, y_2)(t) = -2\cos t \sin t$ . . . . T F

If $y_1(t)$ and $y_2(t)$ are solutions to $y'' + p(t)y' + q(t)y = 0$ in an open interval I, and they have extrema at the same point t_0 in I, they can form a fundamental set of solutions on I . . . . T F

Problem 2. On what interval will the following initial value problem have a unique solution? Explain your reasoning. You should not have to solve the differential equation to answer this question.

$$t\frac{dy}{dt} + 2y = 4t^2, \;\; y(-1) = 2$$

Problem 3. Given the initial value problem

$$y' = y + 1 - t, \quad y(0) = 0,$$

approximate the solution by calculating $\phi_2(t)$ using the method of successive approximations (Picard's iteration method).

Problem 4. Solve the initial value problem:

$$\frac{dy}{dt} + \frac{2}{t}y = 4t, \;\; y(1) = 2, \quad t > 0.$$

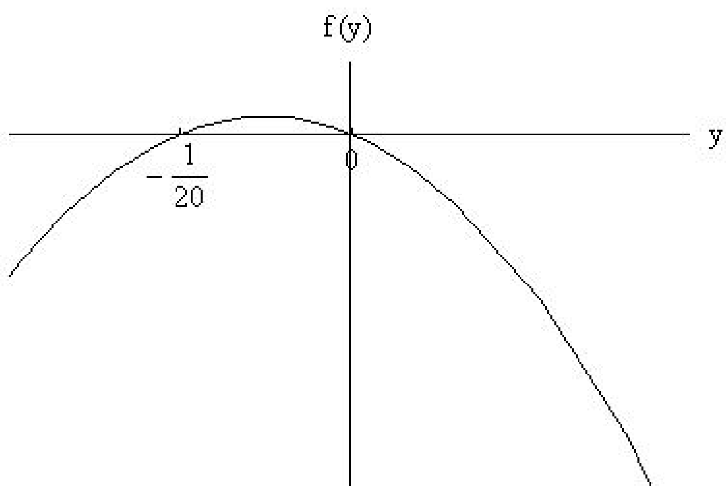
Problem 5. Solve the differential equation (do not worry about where the solution is valid):

$$\frac{dy}{dx} = \frac{-(y^3 - y^2 \sin x - x)}{3xy^2 + 2y\cos x}$$

Problem 6. For the autonomous equation $dy/dt = f(y)$, where a graph of $f(y)$ is given below, construct graphical representations of the solution $y(t)$ vs t. Briefly explain how you found, and indicate the regions where,

- $y(t)$ is increasing or decreasing,
- $y(t)$ is concave up or down.

Also, indicate the equilibrium solutions and whether they are unstable, stable, or semistable.



Problem 7. A large tank (of capacity 700 gallons) is filled with 500 gallons of pure water. Brine containing 2 lb of salt per gallon is pumped into the tank at a rate of 3 gallons per minute. The well-mixed solution is pumped out at the rate of 2 gallons per minute. Derive a differential equation for $Q(t)$, where $Q(t)$ is the amount (in lbs) of salt in the tank at t minutes. Include a initial condition and the time for which the model is valid.
YOU DO NOT HAVE TO SOLVE THE IVP.

Problem 8. Solve the initial value problem:

$$\frac{dx}{dy} = \frac{x^2y^2}{1+x}, \;\; y(1) = 2.$$

Problem 9. Find a general solution of $y''' - 3y'' + 3y' - y = t^2 + 3t$.

Problem 10. For the following logistic equation, construct graphical representations of the solution (explain each step), and identify the critical points as either unstable, stable or semistable solutions. DO NOT SOLVE THE EQUATION EXPLICITLY.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

Problem 11. A large tank is filled with 500 gallons of pure water. Brine containing 2 lb of salt per gallon is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at the same rate.

(a) Show the differential equation in this case is $dQ/dt = 10 - Q/100$, where $Q = Q(t)$ is the number of pounds of salt in the tank at time t.

(b) Solve for $Q(t)$.

Problem 12. Discuss how you could use Picard's Iteration Method to solve the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(0) = 0.$$

Problem 13. Solve the initial value problem:

$$\frac{dy}{dx} + \frac{1}{x}y = 2, \quad y(-1) = 0, \quad x < 0.$$

Problem 14. Consider the autonomous initial value problem

$$\frac{dy}{dt} = f(y) = y^2 - y, \qquad y(0) = y_0.$$

Sketch the function $f(y)$ vs. y and use it to help you construct graphical representations of the solution for a variety of y_0. Clearly explain how you know when the solution $y(t)$ is increasing or decreasing, and when the solution $y(t)$ is concave up or concave down. Determine the equilibrium solutions, and classify them as stable, unstable, or semistable.

Problem 15. A tank of capacity 1500 L contains 1000 L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 L/min. Through another pipe, brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed, and drains from the tank at a rate of 20 L/min. Find the initial value problem which models this situation. You may use $y(t)$ to denote the amount of salt (in kg) in the tank at any time t (in minutes). **You do not have to solve the model you create, just determine what it is.**

Problem 16. A spring–mass system with no forcing function is modelled by a second order linear differential equation. The system parameters are as follows: spring constant $k = 20$ N/m, mass $m = 4$ kg, and (viscous) damping constant $\gamma > 0$. Write down the differential equation associated with this system. What value must γ be for the system to be critically damped?

Problem 17. The differential equation $u'' + \gamma u' + 200u = 0$ is used to model a spring–mass system. What value must γ be for the system to be critically damped?

Problem 18. If the functions $y_1(t)$ and $y_2(t)$ are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, **prove** that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions.

Problem 19. Solve the initial value problem when $\omega \neq \omega_0 = \sqrt{k/m}, k/m > 0$:

$$mu'' + ku = \sin(\omega t), \qquad u(0) = 0, u'(0) = 0$$

Problem 20. Find the general solution to the differential equation

$$y'' + 9y = \frac{1}{4}\csc 3t.$$

Problem 21. Find the general solution (using variation of parameters) of

$$y'' + y = \sec t, \ \ t > 0.$$

Problem 22. Solve the initial value problem

$$mu'' + ku = F_0 \sin(\omega t), \qquad u(0) = 0, u'(0) = 0$$

when $\omega \neq \omega_0 = \sqrt{k/m}$.

Problem 23. The differential equation $5y'' + 2y' + y = 0$ has solutions $y_1 = e^{(-1/5+2i/5)t}$ and $y_2 = e^{(-1/5-2i/5)t}$, which form a fundamental set of solutions. However, these are complex functions. Determine two real valued functions which satisfy the differential equation. **Show your work**. You don't need to show the functions you find form a fundamental set of solutions.

Problem 24. If the functions $y_1(t)$ and $y_2(t)$ are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $y_3 = ay_1 + by_2$ and $y_4 = cy_1 - dy_2$ (a, b, c, d are constants) are also solutions. Under what condition will y_3 and y_4 also form a linearly independent set of solutions?

Problem 25. Use the method of reduction of order to find a second solution to the differential equation (one solution is provided).

$$t^2y'' - 4ty' + 6y = 0, t > 0; \ \ y_1(t) = t^2$$

Problem 26. Solve the initial value problem

$$y'' - 2y' - 3y = t + \frac{2}{3}, \ \ y(0) = 0, \ \ y'(0) = 0.$$

Problem 27. Find a general solution of $y''' - 3y'' + 3y' - y = t^2$.

Problem 28. Find a general solution of $y^{(4)} + 2y^{(2)} + y = t^2$.