General Concepts: First Order Equations

• What is a differential equation and the terminology (order, linear, nonlinear, ordinary, partial, systems, general solution, etc).

• Direction fields (what they are, how to construct them for simple cases, what can we learn from them, etc)

• Infinite families of solutions

• Initial value Problems

• Equilibrium solutions, stable, unstable, semistable

Differences between Linear & Nonlinear Equations

• A *general solution* exists for first order linear when the coefficients are continuous (uniqueness and existence theorems)

A general solution is a family of curves, but a family of curves may not be a general solution.

• There is an explicit solution y for the first order linear equation

• The points of discontinuity are evident without solving the equation for a linear equation

• None of the above are true for nonlinear equations

• intervals on which solutions will exist for first order linear equations (existence and uniqueness theorem)

Techniques

• Separable equations

• Integrating factor technique $\mu(y)$ for linear first order equations

• Exact equations

• Mathematical Modeling (mixing problems)

• Autonomous equations $\frac{dy}{dt} = f(y)$ (Logistic, Critical Threshold), how to construct a graphical representation from the rate function (the details), a solution could become infinite at finite time, how to solve

• Basic integrals, simple parts, simple partial fractions (integration formulas for difficult integrals may be provided)

• Picard's iteration method

Example: Separable

General Form: $G(x)dx = H(y)dy$ or $\dfrac{dy}{dx} = \dfrac{G(x)}{H(y)}$.

Solve the initial value problem: $\dfrac{dy}{dx} = \dfrac{y}{1+x}, \quad y(1) = 1$.

- Note that this can be written as: $\dfrac{dy}{dx} = \dfrac{\left(\frac{1}{y}\right)}{\left(\frac{1}{1+x}\right)}$ or $\dfrac{1}{1+x}dx = \dfrac{1}{y}dy$.

- So it is separable. HINT: Separable equations often require the use of partial fractions!

- Integrate directly to get the solution:

$$\begin{aligned}
\int \frac{1}{1+x}dx &= \int \frac{1}{y}dy \\
\ln|1+x| &= \ln|y| + C \\
1+x &= ye^C \\
y &= C_1(1+x)
\end{aligned}$$

- Apply the initial condition to determine the constant C_1:

$$y(1) = 1 \longrightarrow 1 = C_1(1+1) \longrightarrow C_1 = \frac{1}{2}$$

- So the solution to the initial value problem is: $y = \dfrac{1}{2}(x+1)$.

Example: First Order Linear Equation Variable Coefficients: Integrating Factor (you must show these details in your solution)

General Form: $\dfrac{dy}{dt} + p(t)y = g(t)$.

Solve $\dfrac{dy}{dx} = 2y + x^2 + 5$.

- Rewrite in General Form:

$$\frac{dy}{dx} - 2y = x^2 + 5$$

- Identify that this is first order linear with:

$$p(t) = -2, g(t) = x^2 + 5$$

- Use the integrating factor technique; multiply by $\mu(x)$:

$$\mu(x)\frac{dy}{dx} - 2\mu(x)y = \mu(x)(x^2+5)$$

- Compare the Left Hand Side of above with the product rule expansion of the derivative:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \frac{d\mu(x)}{dx}y$$

- The comparison gives us the differential equation for the integrating factor:

$$\frac{d\mu(x)}{dx} = -2\mu(x)$$

- Solve for the integrating factor (constant of integration does not need to be kept at this point, it will cancel out later):

$$\begin{aligned}
\frac{d\mu}{\mu} &= -2dx \\
\int \frac{d\mu}{\mu} &= \int -2dx \\
\ln|\mu| &= -2x \\
\mu &= e^{-2x}
\end{aligned}$$

- Substitute back:

$$\begin{aligned}
\mu(x)\frac{dy}{dx} - 2\mu(x)y &= \mu(x)(x^2+5) \\
e^{-2x}\frac{dy}{dx} - 2e^{-2x}y &= e^{-2x}(x^2+5) \\
\frac{d}{dx}[e^{-2x}y] &= e^{-2x}(x^2+5) \\
d[e^{-2x}y] &= e^{-2x}(x^2+5)dx \\
\int d[e^{-2x}y] &= \int e^{-2x}(x^2+5)dx \text{ integral provided} \\
e^{-2x}y &= e^{-2x}(\frac{-11}{4} - \frac{x}{2} - \frac{x^2}{2}) + C \\
y &= (\frac{-11}{4} - \frac{x}{2} - \frac{x^2}{2}) + Ce^{2x}
\end{aligned}$$

is the solution of the differential equation.

Example: Exact: nonlinear first order equations

$$\boxed{\text{General Form: } M(x,y) + N(x,y)\frac{dy}{dx} = 0.}$$

Solve $(e^{2y} - y\cos(xy))dx + (2xe^{2y} - x\cos(xy) + 2y)dy = 0$.

• Identify $M(x,y)$ and $N(x,y)$:

$$M(x,y) = e^{2y} - y\cos(xy) \quad N(x,y) = 2xe^{2y} - x\cos(xy) + 2y$$

• Determine the partial derivatives of M and N ("M_y I love exact DEs!"):

$$M_y(x,y) = 2e^{2y} - \cos(xy) + xy\sin(xy) = N_x(e,y)$$

• The above means that this equation is exact. Therefore, there exists a function ψ that satisfies:

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= M(x,y) \\ \frac{\partial \psi}{\partial y} &= N(x,y) \end{aligned}$$

• Pick the first of these equations, and integrate to find an expression for ψ:

$$\begin{aligned} \psi(x,y) &= \int M dx + h(y) \\ &= \int \left(e^{2y} - y\cos(xy)\right) dx + h(y) \\ &= e^{2y}x - \sin(xy) + h(y) \end{aligned}$$

• The solution to the DE is given by $\psi(x,y) = C$, and all that remains to do is determine $h(y)$. We now differentiate the above with respect to y:

$$\frac{\partial \psi}{\partial y} = 2xe^{2y} - x\cos(xy) + \frac{dh(y)}{dy}$$

• Now compare with

$$\frac{\partial \psi}{\partial y} = N(x,y) = 2xe^{2y} - x\cos(xy) + 2y$$

to get a differential equation for $h(y)$:

$$\frac{dh(y)}{dy} = +2y$$

• Solve the DE for $h(y)$ (NOTE: no constant of integration here–it would get folded into the constant C later):

$$\begin{aligned} dh &= 2ydy \\ \int dh &= \int 2ydy \\ h(y) &= y^2 \end{aligned}$$

• Substitute back to obtain the implicit solution to the differential equation as:

$$e^{2y}x - \sin(xy) + y^2 = C$$

General Concepts: Second Order Equations

Homogeneous Equations $ay'' + by' + cy = 0$

- Assume a solution looks like $y = e^{rt}$
- Sub into the DE to get characteristic equation. The roots of the characteristic equation $ar^2 + br + c = 0$ yields the general solutions:
  - two real roots r_1, r_2

    $$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

  - complex conjugate roots $r = \lambda \pm \mu i$

    $$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$$

  - Know how the real valued solutions were found from the complex valued
  - one real root r of multiplicity 2

    $$y(t) = c_1 e^{rt} + c_2 t e^{rt}$$

  - Know how the second solution was found
  - Know how the solutions differ in behaviour (oscillatory, decay, growth)
- Fundamental theory
  - Superposition principle (when can we add solutions to get another solution?)
  - Wronskian $W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$
  - linear independence: Two functions f and g are linearly independent on an interval I if there exist two constants k_1, k_2, not both zero, such that:

    $$k_1 f(t) + k_2 g(t) = 0$$

    for all t in I.
  - Abel's Theorem: $W(y_1, y_2)(t) = c \exp\left(-\int p(t)dt\right)$
  - fundamental set of solutions
  - relationship between linear independence, fundamental sets of solutions, and the Wronskian:
    1. The functions y_1 and y_2 are a fundamental set of solutions on I
    2. The functions y_1 and y_2 are linearly independent on I
    3. $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I
    4. $W(y_1, y_2)(t) \neq 0$ for all t in I
- Reduction of Order to find a second solution when you already have one solution $y_1(t)$ (let $y = v(t)y_1(t)$ and substitute to find second solution)

Nonhomogeneous Equations $ay'' + by' + cy = g(t)$

- Solution is $y(t) = y_c(t) + y_p(t)$ where $y_c(t)$ is the complementary solution (solution of the associated homogeneous equation) and $y_p(t)$ is any particular solution (solution of the nonhomogeneous equation).
- Two methods to determine $y_p(t)$
  - *Undetermined coefficients*, for a nonhomogeneous term of the form exponential, cosine, sine, polynomial.

    If any part of your assumed solution appears in the complementary solution, you will not be able to determine the coefficients. This is because that part will satisfy the homogeneous equation and reduce to zero. If this happens, multiply your entire assumed solution by powers of t until your assumed solution does not contain part of the complementary solution.

    Summary of how to use undetermined coefficients:
    * If $g(t) = e^{\beta t}$, assume particular solution is proportional to $e^{\beta t}$.
    * If $g(t) = \sin \beta t, \cos \beta t$, assume the particular solution is proportional to $A \cos \beta t + B \sin \beta t$.
    * If $g(t)$ is a polynomial, than assume the particular solution is a polynomial of like degree.
    * If $g(t)$ is a product of the above forms, assume the particular solution is the corresponding product. Multiply out and define new constant for each different function of t.
    * If $g(t)$ has more than one term, split the DE up and solve for a particular solution for each term individually.
  - *Variation of parameters*; know how to work the method through from the beginning.

    Assume $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.

    Differentiate to substitute into the differential equation, where you assume $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$.

    The differential equation provides the second equation, and you will have a system of 2 equations in the 2 unknowns $u_1'(t)$ and $u_2'(t)$. Solve using Cramer's rule and you will find

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{W(y_1, y_2)(t)}, \qquad u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{W(y_1, y_2)(t)}$$

    Integrate to determine $u_1(t)$ and $u_2(t)$.
- mechanical and electrical vibrations
  - general behaviour of the solutions (growth, decay, neither)
  - Interpret a given differential equation in terms of a mechanical system
    * importance of obtaining a model that has a unique solution, since the physical model has a unique solution (this leads us to avoid nonlinear models if possible)
  - Unforced Systems
    * undamped free vibrations (harmonic oscillator)
      · equation is $mu'' + ku = 0$.
      · derive the solution $u(t) = A \cos \omega_0 t + B \sin \omega_0 t$, $\omega_0^2 = k/m$.
    * damped free vibrations
      · equation is $mu'' + \gamma u' + ku = 0$.
      · derive the solution, and understand (don't memorize, work from roots of characteristic equation):
      · critical damping $\gamma = 2\sqrt{km}$ (no oscillations) (real root of multiplicity 2)
      · overdamping $\gamma > 2\sqrt{km}$ (no oscillations) (two distinct real valued roots)
      · underdamping $\gamma < 2\sqrt{km}$ (oscillations, not periodic) (complex roots)

General Concepts: nth order linear equations constant coefficients

Homogeneous equations $\sum_{i=0}^{n} c_i y^{(i)} = 0$

- assume solution looks like $y = e^{rt}$
- characteristic equation has roots r_i, $i = 1, 2, 3, \ldots, n$.
  - distinct roots: $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$, etc.
  - if repeated roots $r_1 = r_2 = r_3$; $y_1 = e^{r_1 t}, y_2 = te^{r_1 t}, y_3 = t^2 e^{r_1 t}$, etc.
  - if complex roots $r_{1,2} = \lambda \pm \mu i$; $y_1 = e^{\lambda t}\cos \mu t, y_2 = e^{\lambda t}\sin \mu t$

Nonhomogeneous equations $\sum_{i=0}^{n} c_i y^{(i)} = g(t)$

- solve associated homogeneous equation first.
- undetermined coefficients, assume solution looks like $g(t)$: $Y = At + B$, $Y = e^{\omega t}$, $Y = A\cos \omega t + B \sin \omega t$.
- choice for particular solution should not contain any part of the homogeneous solution
- variation of parameters

Example Solve $y''' - 4y' = t$.

Solve homogeneous equation first, $y''' - 4y' = 0$.

Assume $y = e^{rt}$. Substitute into differential equation to obtain characteristic equation $r^3 - 4r = r(r-2)(r+2) = 0$.

In this case, $y_1 = 1, y_2 = e^{-2t}, y_3 = e^{2t}$. Complementary solution is $y_c(t) = c_1 + c_2 e^{-2t} + c_3 e^{2t}$.

To get particular solution, use undetermined coefficients.

$$
\begin{aligned}
Y &= (At + B) \;\text{ overlap with } y_c \\
&= (At + B)t \;\text{ will work} \\
Y &= At^2 + Bt \\
Y' &= 2At + B \\
Y'' &= 2A \\
Y''' &= 0 \\
y''' - 4y' &= t \\
0 - 4(2At + B) &= t \\
-8At - 4B &= t
\end{aligned}
$$

So $A = -1/8$ and $B = 0$.

The general solution is $y(t) = c_1 + c_2 e^{-2t} + c_3 e^{2t} - t^2/8$.

If initial conditions are given, they could be used at this point to determine the values of c_1, c_2 and c_3.