Questions

Example (2.8.1) Transform the initial value problem $y' = t^2 + y^2$, $y(1) = 2$ into an equivalent problem with the initial point at the origin.

Example (2.8.7) For the initial value problem $y' = ty + 1$, $y(0) = 0$, using Picard's method determine $\phi_n(t)$ and then plot $\phi_n(t)$ for $n = 1, 2, 3, 4$.

Example (2.8.14) Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}, 0 \le x \le 1$.

(a) Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \le x \le 1$, and hence $\displaystyle\int_0^1 \lim_{n\to\infty} \phi_n(x)\, dx = 0$.

(b) Show that $\displaystyle\int_0^1 2nxe^{-nx^2}\, dx = 1 - e^{-n}$, and hence $\displaystyle\lim_{n\to\infty} \int_0^1 \phi_n(x)\, dx = 1$.

Solutions

Example (2.8.1) Transform the initial value problem $y' = t^2 + y^2$, $y(1) = 2$ into an equivalent problem with the initial point at the origin.

The transformation is driven by the initial condition. Define new coordinates:

$$\tilde{y}(\tilde{t}) = y(t) - 2, \tilde{t} = t - 1.$$

The initial conditions for the new coordinates become:

$$\begin{aligned} &\text{if } t = 1, \text{ then } \tilde{t} = 1 - 1 = 0. \\ &\tilde{y}(\tilde{t} = 0) = \tilde{y}(t = 1) = y(1) - 2 = 2 - 2 = 0, \end{aligned}$$

Start with the new equation:

$$\begin{aligned} \tilde{y}(\tilde{t}) &= y(t) - 2 \\ y(t) &= \tilde{y}(\tilde{t}) + 2 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}[y(t)] &= \frac{d}{dt}[\tilde{y}(\tilde{t}) + 2] \\ \frac{dy}{dt} &= \frac{d}{d\tilde{t}}\tilde{y}(\tilde{t}) \cdot \frac{d\tilde{t}}{dt} \\ &= \frac{d\tilde{y}}{d\tilde{t}} \cdot (1) \\ &= \frac{d\tilde{y}}{d\tilde{t}} \end{aligned}$$

The initial value problem therefore becomes:

$$\frac{d\tilde{y}}{d\tilde{t}} = (\tilde{t} + 1)^2 + (\tilde{y} + 2)^2, \tilde{y}(0) = 0.$$

Example (2.8.7) For the initial value problem $y' = ty + 1$, $y(0) = 0$, using Picard's method determine $\phi_n(t)$ and then plot $\phi_n(t)$ for $n = 1, 2, 3, 4$.

Picard's method provides approximate solutions to the initial value problem

$$y' = f(t, y), y(0) = 0,$$

by calculating the sequence of functions

$$\begin{aligned}\phi_0(t) &= 0\\ \phi_{n+1}(t) &= \int_0^t f(s,\phi_n(s))\,ds, n=0,1,2,3,4,\ldots\end{aligned}$$

For this problem, $f(t,y)=ty+1$ and we find:

$$\begin{aligned}\phi_0(t) &= 0\\ \phi_1(t) &= \int_0^t f(s,\phi_0(s))\,ds = \int_0^t f(s,0)\,ds = \int_0^t (s(0)+1)\,ds = t\\ \phi_2(t) &= \int_0^t f(s,\phi_1(s))\,ds = \int_0^t f(s,s)\,ds = \int_0^t (s(s)+1)\,ds = \frac{t^3}{3}+t\\ \phi_3(t) &= \int_0^t f(s,\phi_2(s))\,ds = \int_0^t f(s,\frac{s^3}{3}+s)\,ds = \int_0^t (s(\frac{s^3}{3}+s)+1)\,ds = \frac{t^5}{3\cdot 5}+\frac{t^3}{3}+t\\ \phi_4(t) &= \int_0^t f(s,\phi_3(s))\,ds = \int_0^t f(s,\frac{s^5}{3\cdot 5}+\frac{s^3}{3}+s)\,ds = \int_0^t (s(\frac{s^5}{3\cdot 5}+\frac{s^3}{3}+s)+1)\,ds = \frac{t^7}{3\cdot 5\cdot 7}+\frac{t^5}{3\cdot 5}+\frac{t^3}{3}+t\end{aligned}$$

From this, we can guess the pattern. It looks like we have

$$\phi_n(t) = \sum_{i=1}^{n} \frac{t^{2i-1}}{1\cdot 3\cdot 5\cdots(2i-1)}.$$

The plots are in the *Mathematica* file.

Example (2.8.14) Consider the sequence $\phi_n(x)=2nxe^{-nx^2}, 0\le x\le 1$.

(a) Show that $\lim_{n\to\infty}\phi_n(x)=0$ for $0\le x\le 1$, and hence $\displaystyle\int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx = 0$.

(b) Show that $\displaystyle\int_0^1 2nxe^{-nx^2}\,dx = 1-e^{-n}$, and hence $\displaystyle\lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx = 1$.

In this example, $\displaystyle\lim_{n\to\infty}\int_0^1 \phi_n(x)\,dx \ne \int_0^1 \lim_{n\to\infty}\phi_n(x)\,dx$ even though $\lim_{n\to\infty}\phi_n(x)=0$ exists and is continuous.

(a)

$$\begin{aligned}\lim_{n\to\infty}\phi_n(x) &= \lim_{n\to\infty} 2nxe^{-nx^2}\\ &= 2x\lim_{n\to\infty}\frac{n}{e^{nx^2}} \longrightarrow \frac{\infty}{\infty} \text{ use l'Hospital's rule}\\ &= 2x\lim_{n\to\infty}\frac{\frac{d}{dn}[n]}{\frac{d}{dn}[e^{nx^2}]}\\ &= 2x\lim_{n\to\infty}\frac{1}{x^2e^{nx^2}}\\ &= \frac{2}{x}\lim_{n\to\infty}\frac{1}{e^{nx^2}} = 0 \text{ if } x\ne 0\end{aligned}$$

If $x=0$, we have $\phi_n(0)=2n(0)e^{-0}=0$.

Therefore, $\lim_{n\to\infty}\phi_n(x)=0$ for $0\le x\le 1$.

(b)

$$\begin{aligned}\int_0^1 2nxe^{-nx^2}\,dx &= -\int_0^{-n} e^u\,du \text{ Substitution:} u=-nx^2; du=-2nx\,dx. \text{When } x=1, u=-n; x=0, u=0.\\ &= e^u\big|_{-n}^{0} = 1-e^{-n}\end{aligned}$$

Therefore, $\lim_{n\to\infty} \int_0^1 2nxe^{-nx^2}\,dx = \lim_{n\to\infty}(1-e^{-n}) = 1$.