Questions

Example (3.6.1) Find a particular solution of the differential equation $y'' - 5y' + 6y = 2e^t$. Check your answer using undetermined coefficients.

Example (3.6.4) Find a particular solution of the differential equation $4y'' - 4y' + y = 16e^{t/2}$. Check your answer using undetermined coefficients.

Example (3.6.13) Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of the corresponding homogeneous differential equation for $t^2y'' - 2y = 3t^2 - 1$ for $t > 0$. Then find a particular solution of the nonhomogeneous differential equation.

Solutions

Example (3.6.1) Find a particular solution of the differential equation $y'' - 5y' + 6y = 2e^t$. Check your answer using undetermined coefficients.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$y'' - 5y' + 6y = 0$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 5y' + 6y &= 0 \\ (r^2 - 5r + 6)e^{rt} &= 0 \\ r^2 - 5r + 6 &= 0 \quad \text{characteristic equation} \\ (r-3)(r-2) &= 0 \end{aligned}$$

The roots of the characteristic equation are $r_1 = 3$ and $r_2 = 2$. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1e^{3t} + c_2e^{2t}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants (hence the name, variation of parameters):

$$Y(t) = \sum_{i=1}^{2} \mu_i(t)y_i(t) = \mu_1(t)e^{3t} + \mu_2(t)e^{2t}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)e^{3t} + 3\mu_1(t)e^{3t} + \mu_2'(t)e^{2t} + 2\mu_2(t)e^{2t} = 3\mu_1(t)e^{3t} + 2\mu_2(t)e^{2t}.$$

where we have introduced the condition:

$$\mu_1'(t)e^{3t} + \mu_2'(t)e^{2t} = 0. \qquad (1)$$

Differentiate again:

$$Y''(t) = 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} + 9\mu_1(t)e^{3t} + 4\mu_2(t)e^{2t}.$$

Substitute into the differential equation:

$$\begin{aligned} y'' - 5y' + 6y &= 2e^t \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} + 9\mu_1(t)e^{3t} + 4\mu_2(t)e^{2t} - 5(3\mu_1(t)e^{3t} + 2\mu_2(t)e^{2t}) + 6(\mu_1(t)e^{3t} + \mu_2(t)e^{2t}) &= 2e^t \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} &= 2e^t \qquad (2) \end{aligned}$$

All the terms without derivatives of the $\mu_i(t)$ should drop out at this stage; if they don't check your work! We have the two equations (Eq. (**??**) and (**??**)) in two unknowns:

$$\begin{aligned} \mu_1'(t)e^{3t} + \mu_2'(t)e^{2t} &= 0 \\ 3\mu_1'(t)e^{3t} + 2\mu_2'(t)e^{2t} &= 2e^t \end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 0 & e^{2t} \\ 2e^t & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix}} = \frac{-2e^{3t}}{-e^{5t}} = +2e^{-2t},$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned} \mu_1'(t) &= 2e^{-2t} \\ \int d\mu_1(t) &= 2\int e^{-2t}\, dt \\ \mu_1(t) &= -e^{-2t} + k_1 = -e^{-2t} \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & 2e^t \end{vmatrix}}{\begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix}} = \frac{2e^{4t}}{-e^{5t}} = -2e^{-t},$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned} \mu_2'(t) &= -2e^{-t} \\ \int d\mu_2(t) &= -2\int e^{-t}\, dt \\ \mu_2(t) &= 2e^{-t} + k_2 = 2e^{-t} \text{ set the constant to zero since we are interested in any particular solution} \end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)e^{3t} + \mu_2(t)e^{2t} = -e^{-2t}e^{3t} + 2e^{-t}e^{2t} = e^t.$$

We can use the method of undetermined coefficients to check, since the form of $g(t) = 2e^t$ is one of our special forms.

Initially, assume a solution of the original differential equation is $Y(t) = Ae^t$, since $g(t)$ is an exponential. There is no overlap with $y_c(t)$, so this will be a solution of the nonhomogeneous equation.

All we need to do is substitute it in and determine the value of the constants A and B.

$$\begin{aligned} Y(t) &= Ae^t \\ Y'(t) &= Ae^t \\ Y''(t) &= Ae^t \end{aligned}$$

Substitute into the differential equation

$$\begin{aligned} y'' - 5y' + 6y &= 2e^t \\ Ae^t - 5Ae^t + 6Ae^t &= 2e^t \\ 2A &= 2 \\ A &= 1 \end{aligned}$$

A particular solution is therefore $y_p(t) = Y(t) = Ae^t = e^t$, which verifies what we found using variation of parameters.

Example (3.6.4) Find a particular solution of the differential equation $4y'' - 4y' + y = 16e^{t/2}$. Check your answer using undetermined coefficients.

This is a nonhomogeneous constant coefficient equation. We first solve the associated homogeneous differential equation:

$$4y'' - 4y' + y = 0.$$

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.$$

Substitute into the differential equation:

$$\begin{aligned} 4y'' - 4y' + y &= 0 \\ (4r^2 - 4r + 1)e^{rt} &= 0 \\ 4r^2 - 4r + 1 &= 0 \quad \text{characteristic equation} \\ (2r-1)(2r-1) &= 0 \end{aligned}$$

The root of the characteristic equation is $r = 1/2$ of multiplicity two. A fundamental set of solutions to the associated homogeneous equation is $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 e^{t/2} + c_2 te^{t/2}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants:

$$Y(t) = \sum_{i=1}^{2} \mu_i(t) y_i(t) = \mu_1(t)e^{t/2} + \mu_2(t)te^{t/2}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)e^{t/2} + \mu_2'(t)te^{t/2} + \frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) = \frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right)$$

where we have introduced the condition:

$$\mu_1'(t)e^{t/2} + \mu_2'(t)te^{t/2} = 0. \tag{3}$$

Differentiate again:

$$Y''(t) = \frac{1}{2}\mu_1'(t)e^{t/2} + \mu_2'(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) + \frac{1}{4}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{4}te^{t/2} + e^{t/2}\right).$$

Substitute into the differential equation:

$$\begin{aligned}
4y'' - 4y' + y &= 16e^{t/2} \\
4\left[\frac{1}{2}\mu_1'(t)e^{t/2} + \mu_2'(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right) + \frac{1}{4}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{4}te^{t/2} + e^{t/2}\right)\right] & \\
-4\left[\frac{1}{2}\mu_1(t)e^{t/2} + \mu_2(t)\left(\frac{1}{2}te^{t/2} + e^{t/2}\right)\right] + \left[\mu_1(t)e^{t/2} + \mu_2(t)te^{t/2}\right] &= 16e^{t/2} \\
2e^{t/2}\mu_1' + 4(e^{t/2} + \frac{t}{2}e^{t/2})\mu_2' &= 16e^{t/2} \qquad (4)
\end{aligned}$$

All the terms without derivatives of the $\mu_i(t)$ should drop out at this stage; and they did! This is a good indication that our calculations are correct so far. We have the two equations (Eq. (**??**) and (**??**)) in two unknowns:

$$\begin{aligned}
\mu_1'(t) + t\mu_2'(t) &= 0 \\
2\mu_1'(t) + (4 + 2t)\mu_2'(t) &= 16
\end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 0 & t \\ 16 & 4+2t \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 4+2t \end{vmatrix}} = \frac{-16t}{4} = -4t,$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned}
\mu_1'(t) &= -4t \\
\int d\mu_1(t) &= -4\int t\,dt \\
\mu_1(t) &= -2t^2 + k_1 = -2t^2 \text{ set the constant to zero since we are interested in any particular solution}
\end{aligned}$$

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 4+2t \end{vmatrix}} = \frac{16}{4} = 4,$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned}
\mu_2'(t) &= 4 \\
\int d\mu_2(t) &= 4\int dt \\
\mu_2(t) &= 4t + k_2 = 4t \text{ set the constant to zero since we are interested in any particular solution}
\end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)e^{t/2} + \mu_2(t)te^{t/2} = -2t^2e^{t/2} + 4tte^{t/2} = 2t^2e^{t/2}.$$

We can use the method of undetermined coefficients to check, since the form of $g(t) = 16e^{t^2/2}$ is one of our special forms. I leave this to you.

Example (3.6.13) Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are solutions of the corresponding homogeneous differential equation for $t^2y'' - 2y = 3t^2 - 1$ for $t > 0$. Then find a particular solution of the nonhomogeneous differential equation.

This is a nonhomogeneous <u>variable coefficient</u> equation. We will learn how to solve variable coefficient differential equations using series solutions in Chapter 5. All we need to do is verify that the given functions are solutions.

Substitute y_1 into the differential equation:

$$\begin{aligned} y_1 &= t^2 \\ y_1' &= 2t \\ y_1'' &= 2 \\ t^2y'' - 2y &= t^2(2) - 2(t^2) \\ &= 0 \end{aligned} \tag{5}$$

Therefore, y_1 solves the associated homogeneous differential equation.

Substitute y_2 into the differential equation:

$$\begin{aligned} y_2 &= t^{-1} \\ y_2' &= -t^{-2} \\ y_2'' &= 2t^{-3} \\ t^2y'' - 2y &= t^2(2t^{-3}) - 2(t^{-1}) \\ &= 0 \end{aligned} \tag{6}$$

Therefore, y_2 solves the associated homogeneous differential equation.

The complementary solution is therefore

$$y_c(t) = \sum_{i=1}^{2} c_i y_i(t) = c_1 t^2 + c_2 t^{-1}.$$

To find a particular solution, we assume a solution looks similar to the complementary solution, only with functions which we need to determine instead of constants:

$$Y(t) = \sum_{i=1}^{2} \mu_i(t) y_i(t) = \mu_1(t)t^2 + \mu_2(t)t^{-1}.$$

Now differentiate:

$$Y'(t) = \mu_1'(t)t^2 + \mu_2'(t)t^{-1} + 2\mu_1(t)t - \mu_2(t)t^{-2} = 2\mu_1(t)t - \mu_2(t)t^{-2}$$

where we have introduced the condition:

$$\mu_1'(t)t^2 + \mu_2'(t)t^{-1} = 0. \tag{7}$$

Differentiate again:

$$Y''(t) = 2\mu_1'(t)t - \mu_2'(t)t^{-2} + 2\mu_1(t) + 2\mu_2(t)t^{-3}$$

Substitute into the differential equation:

$$\begin{aligned} t^2y'' - 2y &= 3t^2 - 1 \\ t^2\left[2\mu_1'(t)t - \mu_2'(t)t^{-2} + 2\mu_1(t) + 2\mu_2(t)t^{-3}\right] - 2\left[\mu_1(t)t^2 + \mu_2(t)t^{-1}\right] &= 3t^2 - 1 \\ 2t^3\mu_1' - \mu_2' &= 3t^2 - 1 \end{aligned} \tag{8}$$

We have the two equations (Eq. (**??**) and (**??**)) in two unknowns:

$$\begin{aligned} 2t^3\mu_1' - \mu_2' &= 3t^2 - 1 \\ \mu_1'(t)t^2 + \mu_2'(t)t^{-1} &= 0 \end{aligned}$$

Solve using Cramer's rule:

$$\mu_1'(t) = \frac{\begin{vmatrix} 3t^2 - 1 & -1 \\ 0 & t^{-1} \end{vmatrix}}{\begin{vmatrix} 2t^3 & -1 \\ t^2 & t^{-1} \end{vmatrix}} = \frac{3t - t^{-1}}{2t^2 + t^2} = t^{-1} - \frac{1}{3}t^{-3},$$

so we can solve this differential equation for $\mu_1'(t)$:

$$\begin{aligned} \mu_1'(t) &= t^{-1} - \frac{1}{3}t^{-3} \\ \int d\mu_1(t) &= \int \left(t^{-1} - \frac{1}{3}t^{-3}\right) dt \\ \mu_1(t) &= \ln t + \frac{t^{-2}}{6} + k_1 = \ln t + \frac{t^{-2}}{6} \end{aligned}$$

Where we set the constant k_1 to zero since we are interested in any particular solution and there is no absolute value in logarithm since $x > 0$.

Similarly, for $\mu_2(t)$ we find:

$$\mu_2'(t) = \frac{\begin{vmatrix} 2t^3 & 3t^2 - 1 \\ t^2 & 0 \end{vmatrix}}{\begin{vmatrix} 2t^3 & -1 \\ t^2 & t^{-1} \end{vmatrix}} = \frac{-3t^4 + t^2}{2t^2 + t^2} = -t^2 + \frac{1}{3},$$

so we can solve this differential equation for $\mu_2'(t)$:

$$\begin{aligned} \mu_2'(t) &= -t^2 + \frac{1}{3} \\ \int d\mu_2(t) &= \int \left(-t^2 + \frac{1}{3}\right) dt \\ \mu_2(t) &= -\frac{t^3}{3} + \frac{t}{3} + k_2 = -\frac{t^3}{3} + \frac{t}{3} \end{aligned}$$

A particular solution is therefore:

$$y_p(t) = \mu_1(t)t^2 + \mu_2(t)t^{-1} = \left(\ln t + \frac{t^{-2}}{6}\right)t^2 + \left(-\frac{t^3}{3} + \frac{t}{3}\right)t^{-1} = t^2 \ln t + \frac{1}{2} - \frac{t^2}{3}.$$

Note that $-t^2/3 = cy_1(t)$ is actually a solution of the homogeneous equation. A simpler particular solution is therefore

$$y_p(t) = t^2 \ln t + \frac{1}{2}.$$

We could use the method of undetermined coefficients to check, since the form of $g(t) = 3t^2 - 1$ is one of our special forms.