Questions

Example (3.4.6) Find a general solution of the differential equation \( y'' - 6y' + 9y = 0 \).

Example (3.5.7) Find a general solution of the differential equation \( 4y'' + 17y' + 4y = 0 \).

Example (3.4.11) Solve the initial value problem \( 9y'' - 12y' + 4y = 0 \), \( y(0) = 2 \), \( y'(0) = -1 \). Sketch the graph of the solution and describe the behaviour of the solution as \( t \to \infty \).

Example (3.4.16) Solve the initial value problem \( 9y'' - 12y' + 4y = 0 \), \( y(0) = 2 \), \( y'(0) = b \). Determine the critical value of \( b \) that separates solutions that grow positively from those that eventually grow negatively.

Example (3.4.20) Consider the differential equation \( y'' + 2ay' + a^2y = 0 \). Show that one solution is \( y_1(t) = e^{-at} \) by working through the characteristic equation solution. Then, use Abel’s formula to show a second solution to the differential equation is \( y_2(t) = te^{-at} \).

Example (3.4.23) Find a second solution of the differential equation \( t^2y'' - 4ty' + 6y = 0 \) given one solution is \( y_1(t) = t^2 \).

Solutions

Example (3.4.6) Find a general solution of the differential equation \( y'' - 6y' + 9y = 0 \).

Since this is a constant coefficient differential equation, we assume the solution looks like \( y = e^{rt} \). Then:
\[
y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.
\]

Substitute into the differential equation:
\[
\begin{align*}
y'' - 6y' + 9y & = 0 \\
(r^2 - 6r + 9)e^{rt} & = 0 \\
r^2 - 6r + 9 & = 0 \quad \text{characteristic equation} \\
(r - 3)^2 & = 0
\end{align*}
\]
The root of the characteristic equation is \( r = 3 \) of multiplicity 2.

A fundamental set of solutions is therefore
\[
y_1 = e^{3t}, \quad y_2 = te^{3t}.
\]
The general solution is therefore
\[
y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{3t} + c_2 te^{3t}.
\]

Example (3.5.7) Find a general solution of the differential equation \( 4y'' + 17y' + 4y = 0 \).

Since this is a constant coefficient differential equation, we assume the solution looks like \( y = e^{rt} \). Then:
\[
y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2e^{rt}.
\]

Substitute into the differential equation:
\[
\begin{align*}
4y'' + 17y' + 4y & = 0 \\
(4r^2 + 17r + 4)e^{rt} & = 0 \\
4r^2 + 17r + 4 & = 0 \quad \text{characteristic equation} \\
r & = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
& = \frac{-17 \pm \sqrt{289 - 64}}{8} \\
& = -\frac{1}{4}, -4
\end{align*}
\]
The roots of the characteristic equation are $r_1 = -1/4$, and $r_2 = -4$.

A fundamental set of solutions is therefore

$$y_1 = e^{-t/4}, \quad y_2 = e^{-4t}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{-t/4} + c_2 e^{-4t}.$$

**Example (3.4.11)** Solve the initial value problem $9y'' - 12y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$. Sketch the graph of the solution and describe the behaviour of the solution as $t \to \infty$.

Since this is a constant coefficient differential equation, we assume the solution looks like $y = e^{rt}$. Then:

$$y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitute into the differential equation:

$$9y'' - 12y' + 4y = 0$$

$$(9r^2 - 12r + 4)e^{rt} = 0$$

$9r^2 - 12r + 4 = 0$ characteristic equation

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{144 - 144}}{18}$$

$$= \frac{2}{3}$$

The root of the characteristic equation is $r = 2/3$ of multiplicity 2.

A fundamental set of solutions is therefore

$$y_1 = e^{2t/3}, \quad y_2 = te^{2t/3}.$$

The general solution is therefore

$$y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{2t/3} + c_2 te^{2t/3}.$$

Use the initial conditions to determine the constants:

$$y(t) = c_1 e^{2t/3} + c_2 te^{2t/3}$$

$$y'(t) = c_1 \cdot \frac{2}{3} e^{2t/3} + c_2 e^{2t/3} + c_2 \cdot \frac{2}{3} te^{2t/3}$$

$$y(0) = 2 = c_1$$

$$y'(0) = -1 = c_1 \cdot \frac{2}{3} + c_2$$

The solution is $c_1 = 2$ and $c_2 = -7/3$.

The initial value problem has solution $y(t) = 2e^{2t/3} - \frac{7}{3}te^{2t/3}$.

As $t \to \infty$ the solution decreases without bound, since $y'(t) = -\frac{1}{3}e^{2t/3}(9 + 14t) < 0$ for all values of $t > 0$ (remember, the first derivative tells you if a function is increasing or decreasing).
See the associated Mathematica file for a sketch.

**Example (3.4.16)** Solve the initial value problem \( y'' - y' + y/4 = 0, \ y(0) = 2, \ y'(0) = b. \)

Determine the critical value of \( b \) that separates solutions that grow positively from those that eventually grow negatively.

Since this is a constant coefficient differential equation, we assume the solution looks like \( y = e^{rt} \). Then:

\[
y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.
\]

Substitute into the differential equation:

\[
\begin{align*}
y'' - y' + y/4 &= 0 \\
(r^2 - r + 1/4)e^{rt} &= 0 \\
r^2 - r + 1/4 &= 0 \quad \text{characteristic equation} \\
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{1 \pm \sqrt{1-1}}{2} \\
&= \frac{1}{2}
\end{align*}
\]

The root of the characteristic equation is \( r = 1/2 \) of multiplicity 2.

A fundamental set of solutions is therefore

\[
y_1 = e^{t/2}, \quad y_2 = te^{t/2}.
\]

The general solution is therefore

\[
y(t) = \sum_{i=1}^{2} c_i y_i = c_1 e^{t/2} + c_2 te^{t/2}.
\]

Use the initial conditions to determine the constants:

\[
\begin{align*}
y(t) &= c_1 e^{t/2} + c_2 te^{t/2} \\
y'(t) &= c_1 \cdot \frac{1}{2} e^{t/2} + c_2 e^{t/2} + c_2 \cdot \frac{1}{2} e^{t/2} \\
y(0) &= 2 = c_1 \\
y'(0) &= b = c_1 \cdot \frac{1}{2} + c_2
\end{align*}
\]

The solution is \( c_1 = 2 \) and \( c_2 = b - 1 \).

The initial value problem has solution \( y(t) = 2e^{t/2} + (b-1)te^{t/2} = e^{t/2}[2 + (b-1)t] \).

If \( b > 1 \), the solution will grow positively as \( t \to \infty \). If \( b < 1 \), the solution will grow negatively as \( t \to \infty \). The value of \( b \) which separates the two types of behaviour is \( b = 1 \).

See the Mathematica file for some sketches.

**Example (3.4.20)** Consider the differential equation \( y'' + 2ay' + a^2 y = 0 \). Show that one solution is \( y_1(t) = e^{-at} \) by working through the characteristic equation solution. Then, use Abel’s formula to show a second solution to the differential equation is \( y_2(t) = te^{-at} \).

Since this is a constant coefficient differential equation, we assume the solution looks like \( y = e^{rt} \). Then:

\[
y = e^{rt}, \quad y' = re^{rt}, \quad y'' = r^2 e^{rt}.
\]
Substitute into the differential equation:

\[ y'' + 2ay' + a^2y = 0 \]

\[ (r^2 + 2ar + a^2)e^{rt} = 0 \]

\[ r^2 + 2ar + a^2 = 0 \] characteristic equation

\[ (r + a)^2 = 0 \]

The root of the characteristic equation is \( r = -a \) of multiplicity 2.

One solution of the differential equation is of the form \( y_1(t) = e^{-at} \).

Abel's Theorem tells us \( W(y_1, y_2)(t) = c_1 \exp \left( -\int p(t) \, dt \right) \). Therefore,

\[ W(y_1, y_2)(t) = c_1 \exp \left( -\int 2a \, dt \right) = c_1e^{-2at} = y_1(t)y_2'(t) - y_1'(t)y_2(t) \text{ by definition} \]

Take \( y_1(t) = e^{-at} \), the solution we already know. Taking the derivative and substituting into the differential equation \( c_1e^{-2at} = cy_1(t)y_2'(t) - y_1'(t)y_2(t) \), we arrive at the following first order, linear differential equation in \( y_2 \):

\[ y_2' + ay_2 = c_1e^{-at} \]

This can be solved using the integrating factor technique. Multiply by a function \( \mu = \mu(t) \):

\[ \mu y_2' + a\mu y_2 = \mu c_1e^{-at} \]

Now, we want the following to be true:

\[ \frac{d}{dt} [\mu y_2] = \mu y_2' + \mu' y_2 \quad \text{(by the product rule)} \quad (1) \]

\[ = \mu y_2' + \frac{2\mu}{3} y_2 \quad \text{(the left hand side of our equation)} \quad (2) \]

Comparing Eqs. (1) and (2), we arrive at the differential equation that the integrating factor must solve:

\[ \mu a = \mu' \]

This differential equation is separable, so the solution is

\[ \mu a = \frac{d\mu}{dt} \]

\[ \int a \, dt = \int \frac{d\mu}{\mu} \]

\[ \int a \, dt = \int \frac{d\mu}{\mu} \]

\[ at + c_2 = \ln|\mu| \]

\[ e^{at} e^{c_2} = |\mu| \]

\[ \mu = c_3e^{at} \quad \text{where } c_3 = +e^{c_2} \]
Therefore, the original differential equation becomes

\[
\begin{align*}
\mu y_2' + \mu ay_2 &= \mu c_1 e^{-at} \\
3e^{at} y_2' + 3e^{at} ay_2 &= 3e^{at} c_1 e^{-at} \\
e^{at} y_2' + e^{at} ay_2 &= c_1 \\
\frac{d}{dt}[e^{at} y_2] &= c_1 \\
\int d[e^{at} y_2] &= \int c_1 \, dt \\
e^{at} y_2 &= c_1 t + c_4 \\
y_2 &= c_1 e^{-at} + c_4 e^{-at}
\end{align*}
\]

So a second solution is \( y_2(t) = c_1 e^{-at} + c_4 e^{-at} \). Since we are usually interested in a fundamental set of solutions, which will have no constants and no overlap between them, we can choose a fundamental set of solutions to be:

\( y_1(t) = e^{-at} \), and \( y_3(t) = te^{-at} \).

This verifies the result we saw in class using a completely different method. We you can see things in two different ways that’s a great thing! And on your assignment you will see a third way, which is very exciting indeed.

**Example (3.4.23)** Find a second solution of the differential equation \( t^2 y'' - 4ty' + 6y = 0 \) given one solution is \( y_1(t) = t^2 \).

First, note that this is not a constant coefficient differential equation, so we cannot assume the solution looks like \( y = e^{rt} \). In fact, this is an Euler equation, which we will study in Section 5.5. You might want to try to use a solution like \( y = e^{rt} \) and see what goes wrong as you attempt to get the characteristic equation.

To use reduction of order, we assume a second solution looks like \( y = v(t) y_1(t) = vy_1 = vt^2 \). When you know the second solution, it is a good idea to put it in right away.

\[
\begin{align*}
y(t) &= vt^2 \\
y'(t) &= v't^2 + 2vt \\
y''(t) &= v''t^2 + 4v't + 2v
\end{align*}
\]

Substitute into the differential equation:

\[
\begin{align*}
t^2 y'' - 4ty' + 6y &= 0 \\
t^2(v''t^2 + 4v't + 2v) - 4t(v't^2 + 2vt) + 6(vt^2) &= 0 \\
v''t^4 &= 0 \\
v'' &= 0 \text{ since } t > 0 \\
\frac{dv'}{dt} &= 0 \\
\int d[v'] &= 0 \\
v' + c_1 &= 0 \\
\frac{dv}{dt} &= -c_1 \\
\int dv &= -c_1 \int dt \\
v &= c_2 - c_1 t
\end{align*}
\]

Sometimes you have to use the integrating factor technique to determine \( v \).

A second solution to the differential equation is \( y(t) = vt^2 = c_2 t^2 - c_1 t^3 \).

From this we can identify a fundamental set of solutions as \( y_1(t) = t^2 \) and \( y_2(t) = t^3 \).

A general solution is \( y(t) = k_1 t^2 + k_2 t^3 \).