Questions

Example (4.1.4) Determine the intervals in which solutions are sure to exist for the differential equation $y''' + ty'' + t^2y' + t^3y = \ln t$.

Example (4.1.6) Determine the intervals in which solutions are sure to exist for the differential equation $(x^2 - 4)y^{(iv)} + x^2y''' + 9y = 0$.

Example (4.1.7) Determine whether the functions $f_1(t) = 2t - 3$, $f_2(t) = t^2 + 1$, and $f_3(t) = 2t^2 - t$ are linearly independent or linearly dependent. If they are linearly dependent, find a relation between them.

Example (4.1.11) Verify the functions $f_1(t) = 1$, $f_2(t) = \cos t$, and $f_3(t) = \sin t$ are solutions of the differential equation $y''' + y' = 0$. Determine their Wronskian.

Example (4.2.1) Express $1 + i$ in the form $R(cos\theta + i\sin\theta) = Re^{i\theta}$.

Example (4.2.3) Express $1 - 3$ in the form $R(cos\theta + i\sin\theta) = Re^{i\theta}$.

Example (4.2.4) Express $-i$ in the form $R(cos\theta + i\sin\theta) = Re^{i\theta}$.

Example (4.2.8) Find the square root of the complex number $1 - i$.

Example (4.2.11) Find the general solution of the differential equation $y''' - y'' - y' + y = 0$.

Example (4.2.13) Find the general solution of the differential equation $2y''' - 4y'' - 2y' + 4y = 0$.

Example (4.2.18) Find the general solution of the differential equation $y^{(6)} - y'' = 0$.

Solutions

Example (4.1.4) Determine the intervals in which solutions are sure to exist for the differential equation $y''' + ty'' + t^2y' + t^3y = \ln t$.

The functions $P_1(t) = t$, $P_2(t) = t^2$ and $P_3(t) = t^3$ are continuous on $t \in (-\infty, \infty)$.

$g(t) = \ln t$ is continuous on $t \in (0, \infty)$.

Therefore, a solution exists on the interval $I = (0, \infty)$.

Example (4.1.6) Determine the intervals in which solutions are sure to exist for the differential equation $(x^2 - 4)y^{(iv)} + x^2y''' + 9y = 0$.

Rewrite in standard form: $y^{(iv)} + \dfrac{x^2}{x^2 - 4}y''' + \dfrac{9}{x^2 - 4}y = 0$.

The functions $P_3(t) = \dfrac{x^2}{x^2 - 4}$, $P_6(t) = \dfrac{9}{x^2 - 4}$ are continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

Therefore, solutions are sure to exist on the interval $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

Example (4.1.7) Determine whether the functions $f_1(t) = 2t - 3$, $f_2(t) = t^2 + 1$, and $f_3(t) = 2t^2 - t$ are linearly independent or linearly dependent. If they are linearly dependent, find a relation between them.

$$\begin{aligned}
W(f_1, f_2, f_3)(t) &= \begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} \\
&= \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix} \\
&= (2t-3)\begin{vmatrix} 2t & 4t-1 \\ 2 & 4 \end{vmatrix} - (2)\begin{vmatrix} t^2+1 & 2t^2-t \\ 2 & 4 \end{vmatrix} + (0)\begin{vmatrix} t^2+1 & 2t^2-t \\ 2t & 4t-1 \end{vmatrix} \\
&= (2t-3)(\cancel{8t} - \cancel{8t} + 2) - (2)(\cancel{4t^2} + 4 - \cancel{4t^2} + 4t) \\
&= 4t - 6 - 8 - 8t \\
&= -4t - 14 \neq 0 \text{ for } t = 0
\end{aligned}$$

Therefore, $2t-3$, t^2+1, and $2t^2-t$ are linearly independent.

Example (4.1.11) Verify the functions $f_1(t) = 1$, $f_2(t) = \cos t$, and $f_3(t) = \sin t$ are solutions of the differential equation $y''' + y' = 0$. Determine their Wronskian.

Verify they are solutions via direct substitution:

$$y_1(t) = 1, \quad y_1'(t) = 0, \quad y_1''(t) = 0, \quad y_1'''(t) = 0.$$

$$y''' + y' = 0 + 0 = 0.$$

$$y_2(t) = \cos t, \quad y_2'(t) = -\sin t, \quad y_2''(t) = -\cos t, \quad y_2'''(t) = \sin t.$$

$$y''' + y' = \sin t + (-\sin t) = 0.$$

$$y_3(t) = \sin t, \quad y_3'(t) = \cos t, \quad y_3''(t) = -\sin t, \quad y_3'''(t) = -\cos t.$$

$$y''' + y' = -\cos t + (\cos t) = 0.$$

$$\begin{aligned}
W(1, \cos t, \sin t)(t) &= \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} \\
&= (1)\begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} + 0 \\
&= \sin^2 t + \cos^2 t = 1
\end{aligned}$$

Therefore, 1, $\cos t$, and $\sin t$ are linearly independent.

Examples (4.2.1) (4.2.3) and (4.2.4) See *Mathematica* file.

Example (4.2.8) Find the square root of the complex number $1 - i$.

From the *Mathematica* file we see that we have $1 - i = \sqrt{2}\exp(i(7\pi/4 + 2\pi m))$, $m = \ldots, -2, -1, 0, 1, 2, \ldots$.

$$\begin{aligned}
(1-i)^{1/2} &= \left(\sqrt{2}\exp(i(7\pi/4 + 2\pi m))\right)^{1/2} \\
&= 2^{1/4}\exp(i(7\pi/8 + \pi m)) \\
&= 2^{1/4}\exp(i(7\pi/8 + \pi(0))) \text{ or } 2^{1/4}\exp(i(7\pi/8 + \pi(1))) \qquad \text{other } m \text{ produce these same two values} \\
&= 2^{1/4}\exp(7\pi i/8) \text{ or } 2^{1/4}\exp(15\pi i/8) \\
&= 2^{1/4}\left(\cos(7\pi/8) + i\sin(7\pi/8)\right) \text{ or } 2^{1/4}\left(\cos(15\pi/8) + i\sin(15\pi/8)\right)
\end{aligned}$$

Example (4.2.11) Find the general solution of the differential equation $y''' - y'' - y' + y = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation: $y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}, y''' = r^3e^{rt}$.

$$\begin{aligned} y''' - y'' - y' + y &= 0 \\ r^3e^{rt} - r^2e^{rt} - re^{rt} + e^{rt} &= 0 \\ r^3 - r^2 - r + 1 &= 0 \end{aligned}$$

Now we need the roots of the characteristic equation. We can guess one root, and then factor it out and use the quadratic formula, or use *Mathematica* to get all the roots at once. We find the roots to be $r_1 = -1$, and $r_2 = 1$ of multiplicity 2, that is the characteristic equation can be written as

$$(r-1)^2(r+1) = 0.$$

Therefore, a general solution of the differential equation is $y(t) = c_1e^{-t} + c_2e^t + c_3te^t$.

Example (4.2.13) Find the general solution of the differential equation $2y''' - 4y'' - 2y' + 4y = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y' = re^{rt}, y'' = r^2e^{rt}, y''' = r^3e^{rt}.$$

$$\begin{aligned} 2y''' - 4y'' - 2y' + 4y &= 0 \\ 2r^3e^{rt} - 4r^2e^{rt} - 2re^{rt} + 4e^{rt} &= 0 \\ 2r^3 - 4r^2 - 2r + 4 &= 0 \end{aligned}$$

Now we need the roots of the characteristic equation. We can guess one root, and then factor it out and use the quadratic formula, or use *Mathematica* to get all the roots at once. We find the roots to be $r_1 = 1$, and $r_2 = -1$, and $r_3 = 2$ of multiplicity 1, that is the characteristic equation can be written as

$$(r-1)(r+1)(r-2) = 0.$$

Therefore, a general solution of the differential equation is $y(t) = c_1e^t + c_2e^{-t} + c_3e^{2t}$.

Example (4.2.18) Find the general solution of the differential equation $y^{(6)} - y'' = 0$.

Since the differential equation has constant coefficients and is linear, we assume a solution looks like $y = e^{rt}$. Substitute into the differential equation to get the characteristic equation:

$$y = e^{rt}, y^{(i)} = r^ie^{rt}.$$

$$\begin{aligned} y^{(6)} - y'' &= 0 \\ r^6y^{(6)} - r^2e^{rt} &= 0 \\ r^2(r^4 - 1) &= 0 \\ r^2(r^2 - 1)(r^2 + 1) &= 0 \quad \text{difference of squares} \\ r^2(r-1)(r+1)(r-i)(r+i) &= 0 \end{aligned}$$

Therefore, $r_1 = 1$, and $r_2 = -1$, and $r_3 = i$, and $r_4 = -i$ of multiplicity one, and $r_5 = 0$ of multiplicity two. Therefore, a general solution of the differential equation is $y(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + c_5 + c_6t$.