Ex Solve $y'' + \omega^2 y = g(t) \quad y(0) = 0,\ y'(0) = 1.$

$$\mathcal{L}[y''] + \omega^2 \mathcal{L}[y] = \mathcal{L}[g(t)]$$

From tables: $\mathcal{L}[y] = Y(s)$

$$\mathcal{L}[y''] = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - 1$$

$$\mathcal{L}[g(t)] = G(s)$$

DE becomes:

$$s^2 Y(s) - 1 + \omega^2 Y(s) = G(s)$$

$$Y(s) = \frac{G(s) + 1}{s^2 + \omega^2} = \frac{1}{s^2+\omega^2} + \frac{1}{s^2+\omega^2} \cdot G(s)$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] + \mathcal{L}^{-1}\left[\left\{\frac{1}{s^2+\omega^2}\right\} \cdot G(s)\right]$$

From tables:

$$\mathcal{L}^{-1}[Y(s)] = y(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{1}{\omega}\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{1}{\omega}\sin\omega t$$

$$\mathcal{L}^{-1}\left[\frac{G(s)}{s^2+\omega^2}\right] = \mathcal{L}^{-1}[F(s) \cdot G(s)]$$

$$= (f * g)(t) \quad \text{using convolution theorem.}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{1}{\omega}\sin\omega t$$

$$g(t) = \mathcal{L}^{-1}[G(s)] = g(t)$$

DE becomes:

$$y(t) = \frac{1}{\omega}\sin\omega t + (f * g)(t)$$

$$= \frac{1}{\omega}\sin\omega t + \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \frac{1}{\omega}\sin\omega t + \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\, g(\tau)\, d\tau$$

Ex| $y'' + 3y' + 2y = \cos\alpha t, \quad y(0) = 1 \quad y'(0) = 0.$

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\cos\alpha t]$$

From tables:

$$\mathcal{L}[y] = Y(s)$$

$$\mathcal{L}[y'] = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s$$

$$\mathcal{L}[\cos\alpha t] = \frac{s}{s^2+\alpha^2}$$ (we don't really need to work this out if we plan to use the convolution theorem).

DE becomes:

$$s^2Y(s) - s + 3sY(s) - 3 + 2Y(s) = \frac{s}{s^2+\alpha^2}$$

$$Y(s)(s^2+3s+2) = \frac{s}{s^2+\alpha^2} + s + 3 \quad : \quad s^2+3s+2 = (s+2)(s+1)$$

$$Y(s) = \frac{s}{s^2+\alpha^2} \cdot \frac{1}{(s+2)(s+1)} + \frac{s+3}{(s+2)(s+1)}$$

$$\frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{s+3}{(s+2)(s+1)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+1} - \frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+2} + \frac{2}{s+1} - \frac{1}{s+2}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$$

From tables: $\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$

$$\mathcal{L}^{-1}\left[\frac{2}{s+1}\right] = 2e^{-t}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\right] = \cos\alpha t$$

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+2}\right] = (f*g)(t)$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t \cos(\alpha(t-\tau))e^{-2\tau}\,d\tau$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\alpha^2}\cdot\frac{1}{s+1}\right] = (f*h)(t) \qquad h(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

$$= \int_0^t f(t-\tau)h(\tau)\,d\tau = \int_0^t \cos(\alpha(t-\tau))e^{-\tau}\,d\tau.$$

Solution: $$y(t) = \int_0^t \cos(\alpha(t-\tau))e^{-\tau}\,d\tau - \int_0^t \cos(\alpha(t-\tau))e^{-2\tau}\,d\tau + 2e^{-t} - e^{-2t}$$

$$= \int_0^t \cos(\alpha(t-\tau))\left(e^{-\tau} - e^{-2\tau}\right)d\tau + 2e^{-t} - e^{-2t}$$

You can do the integral using Mathematica.

Ex Solve $y'(t) - \frac{1}{2}\int_0^t (t-\tau)^2 y(\tau)\, d\tau = -t \qquad y(0) = 1.$

Rewrite with convolution:

$$y'(t) - \frac{1}{2} t^2 * y(t)) = -t \qquad y(0) = 1$$

$$\mathcal{L}[y'(t)] - \frac{1}{2}\mathcal{L}[t^2 * y(t)] = -\mathcal{L}[t]$$

From tables:

$$\mathcal{L}[y(t)] = Y(s)$$

$$\mathcal{L}[y'(t)] = sY(s) - y(0) = sY(s) - 1$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}[t^2 * y(t)] = \mathcal{L}[t^2]\mathcal{L}[y(t)]$$ by convolution theorem.

$$= \frac{2}{s^3} Y(s)$$

Equation becomes:

$$sY(s) - 1 - \frac{1}{2}\left(\frac{2}{s^3} Y(s)\right) = -\frac{1}{s^2}$$

$$Y(s)\left(s - \frac{1}{s^3}\right) = 1 - \frac{1}{s^2}$$

$$Y(s)\left(\frac{s^4 - 1}{s^3}\right) = \frac{s^2 - 1}{s^2}$$

$$Y(s) = \frac{s(s^2-1)}{s^4-1} = \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$

$$y(t) = \cos t.$$