Questions

Example (6.1.2) See *Mathematica* file.

Example (6.1.5) Find the Laplace transform of the following functions

(a) t (b) t^2 (c) t^n

Example (6.1.6) Find the Laplace transform of $f(t) = \cos at$, where $a \in \mathbb{R}$.

Finding the inverse Laplace transform requires the use of the table of Laplace transforms in the text. You will be given this table on the final exam, so practice how to use it!

Example (6.2.5) Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+5}$.

Example (6.2.10) Find the inverse Laplace transform of $\dfrac{2s-3}{s^2+2s+10}$.

Example (6.2.27 (c)) The Bessel function of the first kind has the Taylor series $J_0(t) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$.

Assuming the Laplace transforms can be computed term by term, verify that

$$\mathcal{L}[J_0(t)] = (s^2+1)^{-1/2}, \quad s > 1,$$
$$\mathcal{L}[J_0(\sqrt{t})] = s^{-1}e^{-1/(4s)}, \quad s > 0.$$

Solutions

Example (6.1.2) See *Mathematica* file.

Example (6.1.5) Find the Laplace transform of the following functions

(a) t (b) t^2 (c) t^n

(a)

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty te^{-st}\,dt \end{aligned}$$

use parts to do the integral, $\int u\,dv = uv - \int v\,du$: $u = t, dv = e^{-st}\,dt, \quad du = dt, v = -e^{-st}/s$

$$\begin{aligned} &= -\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= -\lim_{t\to\infty} \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\Big|_0^\infty \end{aligned}$$

the limit is zero if $s > 0$ (using l'Hospital's rule)

$$\begin{aligned} &= -0 + \frac{1}{s^2} + 0 \\ \mathcal{L}[t] &= \frac{1}{s^2}, \quad s > 0 \end{aligned}$$

(b) We will use the result from (a) in part (b).

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty t^2 e^{-st}\,dt \end{aligned}$$

use parts to do the integral, $\int u\,dv = uv - \int v\,du$: $u = t^2, dv = e^{-st}\,dt, \quad du = 2tdt, v = -e^{-st}/s$

$$
\begin{aligned}
&= -\frac{t^2}{s}e^{-st}\Big|_0^\infty + \frac{2}{s}\int_0^\infty te^{-st}\,dt \\
&= -\lim_{t\to\infty}\frac{t^2}{s}e^{-st} + \frac{2}{s}\frac{1}{s^2}
\end{aligned}
$$

the limit is zero if $s > 0$ (using l'Hospital's rule twice)

$$
\begin{aligned}
&= -0 + \frac{2}{s^3} \\
\mathcal{L}[t^2] &= \frac{2}{s^3}, \quad s > 0
\end{aligned}
$$

(c) This contains both (a) and (b) for specific values of n.

$$
\begin{aligned}
F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^\infty t^n e^{-st}\,dt
\end{aligned}
$$

use parts to do the integral, $\int u\,dv = uv - \int v\,du$: $u = t^n, dv = e^{-st}\,dt, \quad du = nt^{n-1}dt, v = -e^{-st}/s$

$$
\begin{aligned}
&= -\frac{t^n}{s}e^{-st}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt \\
&= -\lim_{t\to\infty}\frac{t^n}{s}e^{-st} + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt
\end{aligned}
$$

the limit is zero if $s > 0$ (using l'Hospital's rule n times)

$$
\begin{aligned}
&= -0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt \\
&= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt
\end{aligned}
$$

Now, use this result again.

$$
\begin{aligned}
&= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}\,dt \\
&= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\int_0^\infty t^{n-3}e^{-st}\,dt \\
&\vdots \\
&= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{1}{s}\int_0^\infty e^{-st}\,dt \\
&= \frac{n!}{s^n}\int_0^\infty e^{-st}\,dt \\
&= -\frac{n!}{s^n}\cdot\frac{1}{s}e^{-st}\Big|_0^\infty \\
\mathcal{L}[t^n] &= \frac{n!}{s^{n+1}}, \quad s > 0
\end{aligned}
$$

Example (6.1.6) Find the Laplace transform of $f(t) = \cos at$, where $a \in \mathbb{R}$.

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty \cos at\, e^{-st}\, dt \end{aligned}$$

use parts twice to do the integral, $\int u\, dv = uv - \int v\, du$: $u = \cos at, dv = e^{-st}\, dt, \quad du = -a \sin at\, dt, v = -e^{-st}/s$

$$= -\frac{\cos at}{s} e^{-st}\Big|_0^\infty + \frac{a}{s} \int_0^\infty \sin at\, e^{-st}\, dt$$

when evaluating the first term we require $s > 0$

$$= \frac{1}{s} + \frac{a}{s} \int_0^\infty \sin at\, e^{-st}\, dt$$

use parts again, $\int u\, dv = uv - \int v\, du$: $u = \sin at, dv = e^{-st}\, dt, \quad du = a \cos at\, dt, v = -e^{-st}/s$

$$\begin{aligned} &= \frac{1}{s} + \frac{a}{s} \left(-\frac{\sin at}{s} e^{-st}\Big|_0^\infty - \frac{a}{s} \int_0^\infty \cos at\, e^{-st}\, dt \right) \\ &= \frac{1}{s} - \frac{a^2}{s^2} \int_0^\infty \cos at\, e^{-st}\, dt \\ F(s) &= \frac{1}{s} - \frac{a^2}{s^2} F(s) \\ F(s) &= \frac{s}{s^2 + a^2}, \quad s > 0 \\ \mathcal{L}[\cos at] &= \frac{s}{s^2 + a^2}, \quad s > 0 \end{aligned}$$

Finding the inverse Laplace transform requires the use of the table of Laplace transforms in the text.

Example (6.2.5) Find the inverse Laplace transform of $\dfrac{2s+2}{s^2+2s+5}$.

Since the denominator is quadratic, we should complete the square on it and then try to use one of the forms from Table 6.2.1.

$$\begin{aligned} s^2 + 2s + 5 &= s^2 + 2s + 1 - 1 + 5 \\ &= (s+1)^2 + 4 \\ \mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] &= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] \end{aligned}$$

Use Table 6.2.1 # 10 with $a = -1$ and $b = 2$

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t} \cos 2t, \quad s > -1$$

Example (6.2.10) Find the inverse Laplace transform of $\dfrac{2s-3}{s^2+2s+10}$.

Since the denominator is quadratic, we should complete the square on it and then try to use one of the forms from Table 6.2.1.

$$\begin{aligned} s^2+2s+10 &= s^2+2s+1-1+10 \\ &= (s+1)^2+9 \\ \mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] &= \mathcal{L}^{-1}\left[\frac{2s-3}{(s+1)^2+9}\right] \end{aligned}$$

Now, we need to work on the numerator. We want to be able to use Table 6.2.1 # 9 and # 10.

$$\begin{aligned} 2s-3 &= 2\left(s-\frac{3}{2}\right) \\ &= 2\left(s+1-\frac{5}{2}\right) \\ &= 2(s+1)-5 \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] &= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+9}\right]-\mathcal{L}^{-1}\left[\frac{5}{(s+1)^2+9}\right] \\ &= 2\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+9}\right]-\frac{5}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+9}\right] \\ &= 2e^{-t}\cos 3t-\frac{5}{3}e^{-t}\sin 3t \end{aligned}$$

Example (6.2.27 (c)) The Bessel function of the first kind has the Taylor series $J_0(t)=\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$.

Assuming the Laplace transforms can be computed term by term, verify that

$$\begin{aligned} &\mathcal{L}[J_0(t)] = (s^2+1)^{-1/2}, \quad s>1, \\ &\mathcal{L}[J_0(\sqrt{t})] = s^{-1}e^{-1/(4s)}, \quad s>0. \end{aligned}$$

$$\begin{aligned} \mathcal{L}[J_0(t)] &= \int_0^\infty e^{-st}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}\,dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\int_0^\infty t^{2n}e^{-st}\,dt \end{aligned}$$

This integral can be worked out using parts $2n$ times (similar to how Homework 6.1.5(c) was done), or you can use *Mathematica* to determine:

$$\int_0^\infty t^{2n}e^{-st}\,dt = \frac{(2n)!}{s^{2n+1}}, \quad s>0.$$

Therefore, we have

$$\mathcal{L}[J_0(t)] = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2 s^{2n+1}}$$

We can use *Mathematica* to help us recognize this Taylor series.

$$\mathcal{L}[J_0(t)] = \frac{1}{\sqrt{1+s^2}}$$

For the second part of the question, the we get the following:

$$\begin{aligned}
\mathcal{L}[J_0(\sqrt{t})] &= \int_0^\infty e^{-st} \sum_{n=0}^\infty \frac{(-1)^n t^n}{2^{2n}(n!)^2}\, dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{2^{2n}(n!)^2} \int_0^\infty t^n e^{-st}\, dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n n!}{2^{2n}(n!)^2 2^{n+1}} \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n! 2^{2n} s^{n+1}} \\
&= \frac{1}{s} e^{-1/(4s)}
\end{aligned}$$

which is again arrived at with some assistance from *Mathematica*.