7.6.9 $x' = Ax$; $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

Assume $x = \xi e^{\lambda t}$; $x' = \lambda \xi e^{\lambda t}$

Substitute into DE: $\lambda \xi e^{\lambda t} = A \xi e^{\lambda t}$

Characteristic equation: $(A - \lambda I)\xi = 0$

We need the eigenvalues and eigenvectors of the characteristic equation

<u>Get eigenvalues</u>: $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{pmatrix} = 0$$

$$-(1-\lambda)(3+\lambda) + 5 = 0$$
$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{4-8}}{2}$$
$$= -1 \pm i$$

$\rightarrow$ $\lambda^{(1)} = -1 - i$, $\lambda^{(2)} = -1 + i$

and we have complex eigenvalues.

7.6.9 (continued) One eigenvector is all we need, since A is $\mathbb{R}$, we have $x^{(1)} = \bar{x}^{(1)}$.

For $\lambda = \lambda^{(1)} = -1-i$: $(A - \lambda^{(1)} I)\xi^{(1)} = 0$

$$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$$\to \quad \begin{aligned} (2+i)\xi_1 - 5\xi_2 &= 0 \\ \xi_1 + (-2+i)\xi_2 &= 0 \end{aligned}$$

These equations are the same (multiply 2ND equation by $(2+i)$). So we have one equation and two unknowns.

ξ_2 is arbitrary; $\xi_2 = 1 \to \xi_1 = 2-i$

A complex valued solution to the DE is

$$x^{(1)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-(1+i)t}$$

Aside:
Since $x^{(2)} = \bar{x}^{(1)}$, we know $x^{(2)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-(1-i)t}$

7.6.9 (continued) We want $\mathbb{R}$ valued solutions, so let's find the real and imaginary parts of $x^{(1)}$. These will be $\mathbb{R}$, and will satisfy the DE, providing our two $\mathbb{R}$ valued solutions.

$$x^{(1)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-(1+i)t}$$

$$= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \left[e^{-t}(\cos t - i\sin t) \right]$$

$$= e^{-t}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t \right]$$

$$+ ie^{-t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t \right]$$

$$= e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t}\begin{pmatrix} -\cos t - 2\sin t \\ \sin t \end{pmatrix}$$

Two linearly independent solutions of the DE are

$$x^{(1)} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} e^{-t}$$

$$x^{(2)} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} e^{-t}$$

which are $\mathbb{R}$ valued. The general solution is

$$x = c_1 x^{(1)} + c_2 x^{(2)}.$$

7.6.9 (continued)

Use the I.C. to determine the constants:

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is equivalent to

$$1 = 2c_1 + c_2$$
$$1 = c_1$$

$\Rightarrow$ $c_1 = 1$, $c_2 = -1$, and the solution to the IVP is

$$x = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} e^{-t} - \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} e^{-t}$$
$$= e^{-t} \begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix}$$

As $t \to \infty$ $e^{-t} \to 0$. The cosine and sine will oscillate, so the solution will spiral towards zero as $t \to \infty$.