7.8.9 $x' = Ax \quad x(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 3/2 \\ -3/2 & -1 \end{pmatrix}$

Assume $x = \xi e^{\lambda t}$; $x' = \lambda \xi e^{\lambda t}$

Substitute: $\lambda \xi e^{\lambda t} = A \xi e^{\lambda t}$

Characteristic Equation: $(A - \lambda I)\xi = 0$

Get the eigenvalues & eigenvectors:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 3/2 \\ -3/2 & -1-\lambda \end{vmatrix} = 0$$

$$-(1+\lambda)(2-\lambda) + \frac{9}{4} = 0$$

$$\lambda^2 - \lambda + \frac{1}{4} = 0$$

$$(\lambda - 1/2)^2 = 0$$

eigenvalues are $\lambda = \lambda^{(1)} = \frac{1}{2}$ of multiplicity two.

7.8.9 (continued) get the eigenvectors:

$$(A - \lambda^{(1)} I)\xi^{(1)} = 0$$

$$\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix}\begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = 0$$

$\rightarrow \quad \xi_1^{(1)} + \xi_2^{(1)} = 0$ 1 equation, 2 unknowns.

$\xi_1^{(1)}$ arbitrary ; $\xi_1^{(1)} = 1$; $\xi_2^{(1)} = -\xi_1^{(1)} = -1$

$\Rightarrow$ a solution to DE is $x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$

For a second solution, we assume

$$x = \eta^{(1)} t e^{t/2} + \eta^{(0)} e^{t/2}$$ $\left.\begin{matrix}\eta^{(1)} \\ \eta^{(2)}\end{matrix}\right\}$ to be determined.

$$x' = \eta^{(1)} e^{t/2} + \frac{1}{2}\eta^{(1)} t e^{t/2} + \frac{1}{2}\eta^{(0)} e^{t/2}$$

Substitute into the DE:

$$\eta^{(1)} e^{t/2} + \frac{1}{2}\eta^{(1)} t e^{t/2} + \frac{1}{2}\eta^{(0)} e^{t/2} = A\eta^{(1)} t e^{t/2} + A\eta^{(0)} e^{t/2}.$$

7.8.9 (continued) collect powers of t, after cancelling the $e^{t/2}$ (each coefficient must be zero for our assumed solution to be a solution):

t^0: $\quad \eta^{(1)} + \frac{1}{2}\eta^{(0)} - A\eta^{(0)} = 0$

t^1: $\quad \frac{1}{2}\eta^{(1)} - A\eta^{(1)} = 0$

$$\Rightarrow \quad (A - \tfrac{1}{2}I)\eta^{(1)} = 0 \qquad (1)$$

$$(A - \tfrac{1}{2}I)\eta^{(0)} = \eta^{(1)} \qquad (2)$$

(1) is just our original eigenvalue problem; therefore $\eta^{(1)} = \xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Substitute into (2) and solve for $\eta^{(0)}$:

$$\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix} \eta^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\rightarrow \quad 3\eta_1^{(0)} + 3\eta_2^{(0)} = 2$$ 1 eq$^{\underline{n}}$ 2 unknowns.

$\eta_1^{(0)}$ arbitrary; assume $\eta_1^{(0)} = 1$; $\eta_2^{(0)} = -1/3$

$\therefore$ a second solution is

$$x^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t/2} + \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} e^{t/2}$$

The general solution is

7.8.9 (continued)

$$X = c_1 X^{(1)} + c_2 X^{(2)}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{t/2} + \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} e^{t/2} \right]$$

use the initial conditions to determine the constants:

$$X(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1/3 \end{pmatrix}$$

which corresponds to the system

$$3 = c_1 + c_2$$
$$-2 = -c_1 - c_2/3$$

Solve using Cramer's rule:

$$c_1 = \frac{\begin{vmatrix} 3 & 1 \\ -2 & -1/3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1 & -1/3 \end{vmatrix}} = \frac{-1+2}{\frac{-1}{3}+1} = \frac{1}{2/3} = 3/2$$

$$c_2 = \frac{\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix}}{2/3} = \frac{-2+3}{2/3} = \frac{3}{2}$$

The IVP solution is:

$$X = \frac{3}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2} + \frac{3}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{t/2} + \frac{3}{2} \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} e^{t/2}$$

7.8.9 (continued)

$$x = \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{t/2} + \frac{3}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{t/2}$$

$$x_1 = 3e^{t/2} + \frac{3}{2} t e^{t/2} = 3e^{t/2}\left(1+\frac{t}{2}\right)$$

Plot solution in $x_1 x_2$- plane; and $x_1(t)$.

7.8.13 $tx' = Ax \qquad A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

This looks like an Euler type equation.
So let's assume $x = \xi t^{\lambda} \quad x' = \lambda \xi t^{\lambda-1}$

Substitute: $t \lambda \xi t^{\lambda-1} = A \xi t^{\lambda}$

Characteristic Equation: $(A - \lambda I)\xi = 0$

We need the eigenvalues and eigenvectors of the matrix A:

get eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$-(1+\lambda)(3-\lambda) + 4 = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$

$\lambda^{(1)} = 1$ is an eigenvalue of multiplicity 2.

The eigenvalue will satisfy $(A - \lambda^{(1)} I)\xi^{(1)} = 0$.

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

7.8.13 (continued) $\rightarrow \xi_1 - 2\xi_2 = 0$ one equation two unknowns

ξ_1 arbitrary; take $\xi_1 = 2 \rightarrow \xi_2 = \frac{\xi_1}{2} = 1$

One solution of the DE is $x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t$

In our previous study of Euler equations, we could try to find a second solution using reduction of order. Let's try to do something similar here:

Assume $x = \underbrace{\eta^{(1)}}_{} \, v(t) \, t + \eta^{(0)} t$

$v(t)$: some scalar to be determined

t, t: got this from $x^{(1)}$

$\eta^{(1)} = \begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \end{pmatrix} \quad \eta^{(0)} = \begin{pmatrix} \eta_1^{(0)} \\ \eta_2^{(0)} \end{pmatrix}$ are constants to be determined

$$x' = \eta^{(1)} v' t + \eta^{(1)} v + \eta^{(0)}$$

Substitute into the DE:

$$\eta^{(1)} v' t^2 + \eta^{(1)} v t + \eta^{(0)} t = A\eta^{(1)} v t + A\eta^{(0)} t$$

7.8.13 (continued)

$$-\eta^{(1)} v' t + v(A-I)\eta^{(1)} + A\eta^{(0)} = \eta^{(0)}$$

$$v(A-I)\eta^{(1)} + (A-I)\eta^{(0)} = \eta^{(1)} v' t$$

<u>IF</u> $\eta^{(1)} = \xi^{(1)}$, then $(A-I)\eta^{(1)} = 0$

since $\xi^{(1)}$ is the eigenvector associated with eigenvalue 1. Let's try $\eta^{(1)} = \xi^{(1)}$

$$\eta^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow (A-I)\eta^{(0)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} v' t$$

$$\begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix} \begin{pmatrix} \eta_1^{(0)} \\ \eta_2^{(0)} \end{pmatrix} = \begin{pmatrix} 2v't \\ v't \end{pmatrix}$$

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \eta_1^{(0)} \\ \eta_2^{(0)} \end{pmatrix} = \begin{pmatrix} 2v't \\ v't \end{pmatrix}$$

$$\eta_1^{(0)} - 2\eta_2^{(0)} = v't$$

$\Rightarrow$ 1 equation 3 unknowns!

Pick $\eta_1^{(0)} = 1$; $\eta_2^{(0)} = 0$

$$v't = 1$$

$$\int dv = \int \frac{dt}{t} \Rightarrow v = \ln t$$

7.8.13 (continued)

$\therefore$ a second solution is

$$x^{(2)} = \eta^{(1)} v t + \eta^{(0)} t$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$$

The general solution is

$$x = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t \ln t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right]$$

Note: You can get other solutions.

if $\eta_1^{(0)} = 0$ $\eta_2^{(0)} = 1$

$$-2 = v' t$$

$$\int dv = \int -2 \frac{dt}{t}$$

$$v = -2 \ln t$$

and $x^{(3)} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} t \ln t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$