7.5.16 $x' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} x; \; x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Assume $x = \xi e^{\lambda t}$; $x' = \lambda \xi e^{\lambda t}$

substitute into DE $\lambda \xi e^{\lambda t} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \xi e^{\lambda t}$

characteristic equation $(A - \lambda I)\xi = 0$

where $A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$. We need to solve the characteristic equation, that is, determine the eigenvalues and eigenvectors.

get eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$-(2+\lambda)(4-\lambda) + 5 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$\rightarrow \lambda^{(1)} = 3 \quad \lambda^{(2)} = -1$

7.5.16 (continued) Get eigenvectors:

For $\lambda = \lambda^{(1)} = 3$: $(A - \lambda^{(1)} I)\xi^{(1)} = 0$

$$\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$\rightarrow \quad -5\xi_1 + \xi_2 = 0$ One equation, two unknown:

$\xi_1 = 1$ is arbitrary; $\xi_2 = 5\xi_1 = 5$

$\rightarrow \quad x^{(1)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$ is one solution.

For $\lambda = \lambda^{(2)} = -1$: $(A - \lambda^{(2)} I)\xi^{(2)} = 0$

$$\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

$\rightarrow \quad -\xi_1 + \xi_2 = 0$ One equation, two unknow

$\xi_1 = 1$ is arbitrary; $\xi_2 = \xi_1 = 1$.

$\rightarrow \quad x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ is a second solution.

The general solution of the DE is

$$x = c_1 x^{(1)} + c_2 x^{(2)}$$

$$= c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

7.5.16 (continued) Apply the I.C. to determine c_1 & c_2:

$$x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This is equivalent to
$$1 = c_1 + c_2$$
$$3 = 5c_1 + c_2$$

use Cramer's Rule to solve:

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix}} = \frac{1-3}{1-5} = \frac{-2}{-4} = 1/2$$

$$c_2 = \frac{\begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix}}{-4} = \frac{3-5}{-4} = \frac{-2}{-4} = 1/2$$

Note: all these solutions are ok.

The IVP solution is $x = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$

$$= \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix} e^{3t} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} e^{-t}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{3t} + e^{-t}) \\ \frac{1}{2}(5e^{3t} + e^{-t}) \end{pmatrix}$$

A $t \to \infty$, $x \to \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$

so $x_2 \to 5x_1$ as $t \to \infty$.

(0,0) is an asymp. unstable ~~node~~ saddle point
(one e-value $\mathbb{R}>0$
one $\mathbb{R}<0$)