9.1.1 $\quad \dfrac{d\underline{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \underline{x}$

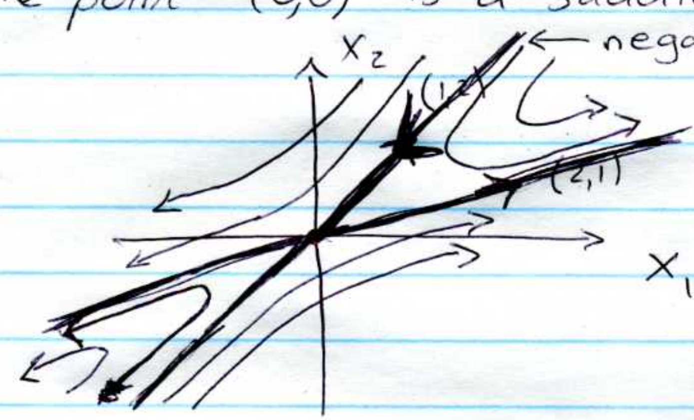
Eigenvalues: $\det(\underline{A} - \lambda \underline{I}) = 0$

$$\det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(-2-\lambda) + 4 = 0$$

$$(\lambda+1)(\lambda-2) = 0 \Rightarrow \lambda = -1, \ \lambda = 2$$

The point $(0,0)$ is a saddle. It is unstable.



Eigenvectors: $\lambda = -1$. Solve $(\underline{A} + 1\underline{I})\underline{\xi} = \underline{0}$

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

$\lambda = 2$. Solve $(\underline{A} - 2\underline{I})\underline{\xi} = \underline{0}$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \underline{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

Solution: $\underline{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$

9.1.3 $\quad \dfrac{d}{dt}\underline{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underline{x}$

Eigenvalues $\quad \det(\underline{A} - \lambda \underline{I}) = 0$

$$\det\begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(-2-\lambda) + 3 = 0$$

$$-4 + \cancel{2\lambda} - \cancel{2\lambda} + \lambda^2 + 3 = 0 \quad \Rightarrow \lambda = \pm 1 \quad \cancel{\lambda = 1,\ \text{multiplicity } 2}$$

Eigenvectors $\lambda = 1$. Solve $(\underline{A} - 1\underline{I})\underline{\xi} = \underline{0}$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

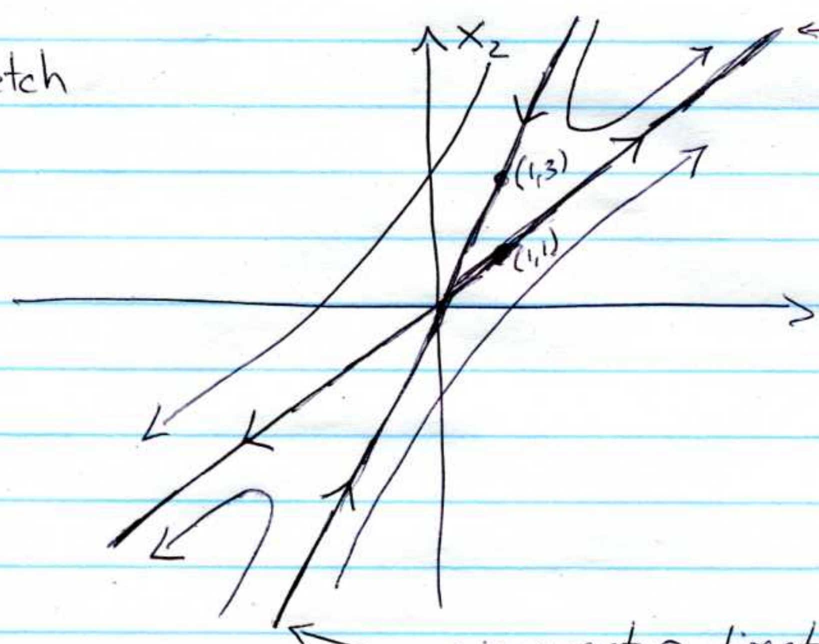
$$\Rightarrow \underline{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underline{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$$

$\lambda = -1$. Solve $(\underline{A} + \underline{I})\underline{\xi} = \underline{0}$

$$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \underline{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

sketch



eigen~~value~~vector direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; eiginvalue > 0 so curves approach this line as $t \to \infty$

eigenvector direction $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$; eigenvalue < 0 so curves approach zero along this direction.

$\underline{9.1.10}$ $\quad \dfrac{d\underline{x}}{dt} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \underline{x}$

<u>Eigenvalues</u> $\quad \det(\underline{A} - \lambda \underline{I}) = 0$

$$\det \begin{pmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) + 10 = 0$$

$$-1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 + 10 = 0 \quad \Rightarrow \quad \lambda = \pm 3i$$

The point (0,0) is a stable center.

<u>Eigenvectors</u>

$\lambda = +3i$: solve $\quad (\underline{A} - 3i\,\underline{I})\,\underline{\xi} = \underline{0}$

$$\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

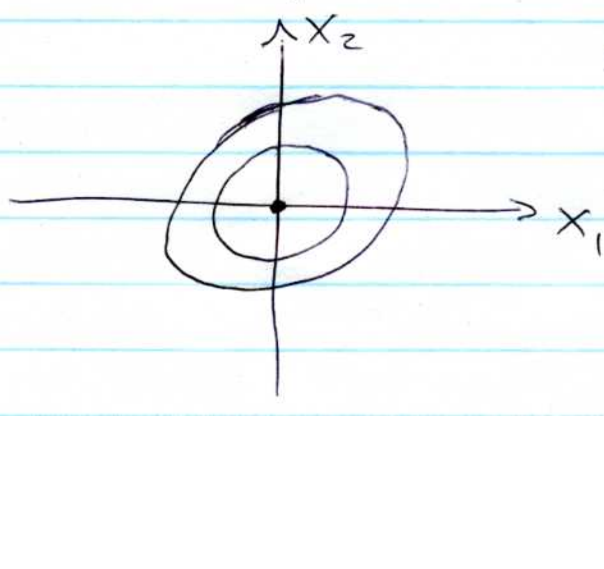
$$\Rightarrow \quad (1-3i)\,\xi_1 + 2\,\xi_2 = 0 \quad \Rightarrow \quad \underline{\xi} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix}$$

$$\underline{x}^{(1)} = \begin{pmatrix} -2 \\ 1-3i \end{pmatrix} e^{3it} = \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix} i \right] \left[\cos 3t + i \sin 3t \right]$$

$$= \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t}_{\underline{x}^{(1)}} \quad + \; i \underbrace{\left[\begin{pmatrix} 0 \\ -3 \end{pmatrix} \cos 3t + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin 3t \right]}_{\underline{x}^{(2)}}$$

$\underline{x}^{(1)} \longleftarrow$ two real-valued solutions $\longrightarrow \underline{x}^{(2)}$

$$\underline{x}^{(1)} = \begin{pmatrix} -2\cos 3t \\ \cos 3t + 3\sin 3t \end{pmatrix} \qquad \underline{x}^{(2)} = \begin{pmatrix} -2\sin 3t \\ -3\cos 3t + \sin 3t \end{pmatrix}$$



solutions will be ellipses.